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Discrete Symmetries and Clifford Algebras

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An algebraic description of basic discrete symmetries (space reversal P, time reversal T, and their combination PT) is studied. Discrete subgroups of orthogonal groups of multidimensional spaces over the fields of real and complex numbers are considered in terms of fundamental automorphisms of Clifford algebras. In accordance with a division ring structure, a complete classification of automorphism groups is established for the Clifford algebras over the field of real numbers. The correspondence between eight double coverings (Dąbrowski groups) of the orthogonal group and eight types of the real Clifford algebras is defined with the use of isomorphisms between the automorphism groups and finite groups. Over the field of complex numbers there is a correspondence between two nonisomorphic double coverings of the complex orthogonal group and two types of complex Clifford algebras. It is shown that these correspondences associate with a well-known Atiyah–Bott–Shapiro periodicity. Generalized Brauer–Wall groups are introduced on the extended sets of the Clifford algebras. The structure of the inequality between the two Clifford–Lipschitz groups with mutually opposite signatures is elucidated. The physically important case of the two different double coverings of the Lorentz groups is considered in details.

1. INTRODUCTION

In 1909, Minkowski showed that a causal structure of the world is described by a 4-dimensional pseudo-Euclidean geometry. In accordance with Minkowski (1909), the quadratic form $x^2 + y^2 + z^2 - c^2t^2$ remains invariant under the action of linear transformations of the four variables x, y, z, and t, which form a general Lorentz group G. As known, the general Lorentz group G consists of an own Lorentz group G_0 and three reflections (discrete transformations) P, T, and PT, where P and T are space and time reversal, and PT is a so-called full reflection. The discrete transformations P, T, and PT added to an identical transformation form a finite group. Thus, the general Lorentz group may be represented by a semidirect product $G_0 \odot \{1, P, T, PT\}$. Analogously, an orthogonal group $O(p, q)$ of the real space $\mathbb{R}^{p,q}$ is represented by the semidirect product of a connected component $O_0(p, q)$ and a discrete subgroup.

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Further, a double covering of the orthogonal group $O(p,q)$ is a Clifford–Lipschitz group $\mathbf{Pin}(p,q)$ which is completely constructed within a Clifford algebra $C\ell_{p,q}$. In accordance with squares of elements of the discrete subgroup ($a = P^2, b = T^2, c = (PT)^2$), there exist eight double coverings (Dąbrowski groups, Dąbrowski, 1988) of the orthogonal group defined by the signatures (a,b,c), where $a,b,c \in \{-,+\}$. Such in brief is a standard description scheme of the discrete transformations.

However, in this scheme there is one essential flaw. Namely, the Clifford–Lipschitz group is an intrinsic notion of the algebra $C\ell_{p,q}$ (a set of all invertible elements of $C\ell_{p,q}$), whereas the discrete subgroup is introduced into the standard scheme in an external way, and the choice of the signature (a,b,c) of the discrete subgroup is not determined by the signature of the space $\mathbb{R}^{p,q}$. Moreover, it is suggested by default that for any signature (p,q) of the vector space, there exist all eight kinds of the discrete subgroups.

In the recent paper Varlamov (1999), to assimilate the discrete transformations into an algebraic framework, it has been shown that elements of the discrete subgroup correspond to fundamental automorphisms of the Clifford algebras. The set of the fundamental automorphisms added to an identical automorphism forms a finite group, for which in virtue of the Wedderburn–Artin theorem there exists a matrix representation. The main subject of Varlamov (1999) is the study of the homomorphism $\mathbb{C}_{n+1} \to \mathbb{C}_n$ and its application in physics, where $\mathbb{C}_n$ is a Clifford algebra over the field of complex numbers $\mathbb{F} = \mathbb{C}$.

The main goal of the present paper is a more explicit and complete formulation (in accordance with a division ring structure of the algebras $C\ell_{p,q}$) of the preliminary results obtained in Varlamov (1999). The classification of automorphism groups of Clifford algebras over the field of real numbers $\mathbb{F} = \mathbb{R}$ and a correspondence between eight Dąbrowski $\mathbf{Pin}^{a,b,c}$ coverings of the group $O(p,q)$ and eight types of $C\ell_{p,q}$ are established in Section 3. It is shown that the division ring structure of $C\ell_{p,q}$ imposes hard restrictions on the existence and choice of the discrete subgroup, and the signature (a,b,c) depends upon the signature of the underlying space $\mathbb{R}^{p,q}$. On the basis of obtained results, a nature of the inequality $\mathbf{Pin}(p,q) \not\simeq \mathbf{Pin}(q,p)$ is elucidated in Section 4. As known, the Lorentz groups $O(3,1)$ and $O(1,3)$ are isomorphic, whereas their double coverings $\mathbf{Pin}(3,1)$ and $\mathbf{Pin}(1,3)$ are nonisomorphic. With the help of Maple V package CLIFFORD (Abłamowicz, 1996, 2000), a structure of the inequality $\mathbf{Pin}(3,1) \not\simeq \mathbf{Pin}(1,3)$ is considered as an example that is, all the possible spinor representations of a Majorana algebra $C\ell_{3,1}$ and a spacetime algebra $C\ell_{1,3}$, and corresponding automorphism groups, are analysed in detail. In connection with this, it should be noted that the general Lorentz group is a basis for (presently most profound in both mathematical and physical viewpoints) Wigner's definition of an elementary particle as an irreducible representation of the inhomogeneous Lorentz group (Wigner, 1964).

It is known that the Clifford algebras are modulo 8 periodic over the field of real numbers and modulo 2 periodic over the field of complex numbers (Ativah–Bott-Shapiro periodicity, Atiyah *et al.*, 1964). In virtue of this periodicity, a structure of the Brauer–Wall group (Budimich and Trautman, 1988; Lounesto, 1997; Wall, 1964) is defined on the set of the Clifford algebras, where a group element is $C\ell$, and a group operation is a graded tensor product. The Brauer–Wall group over the field $\mathbb{F} = \mathbb{R}$ is isomorphic to a cyclic group of the eighth order, and over the field $\mathbb{F} = \mathbb{C}$ to a cyclic group of the second order. Generalizations of the Brauer–Wall groups are considered in Section 5. The Trautman diagrams of the generalized groups are defined as well.

2. PRELIMINARIES

In this section, we will consider some basic facts about Clifford algebras and Clifford–Lipschitz groups, which we will widely use below. Let $\mathbb{F}$ be a field of characteristic 0 ($\mathbb{F} = \mathbb{R},\ \mathbb{F} = \mathbb{C}$), where $\mathbb{R}$ and $\mathbb{C}$ are the fields of real and complex numbers, respectively. A Clifford algebra $C\ell$ over a field $\mathbb{F}$ is an algebra with 2^n basis elements: $\mathbf{e}_0$ (unit of the algebra) $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ and products of the one-index elements $\mathbf{e}_{i_1 i_2 \cdots i_k} = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_k}$. Over the field $\mathbb{F} = \mathbb{R}$, the Clifford algebra denoted as $C\ell_{p,q}$, where the indices p and q correspond to the indices of the quadratic form

$$Q = x_1^2 + \cdots + x_p^2 - \cdots - x_{p+q}^2$$

of a vector space V associated with $C\ell_{p,q}$. The multiplication law of $C\ell_{p,q}$ is defined by the following rule:

$$\mathbf{e}_i^2 = \sigma(q-i)\mathbf{e}_0, \qquad \mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i, \tag{1}$$

where

$$\sigma(n) = \begin{cases} -1 & \text{if } n \leq 0, \\ +1 & \text{if } n > 0. \end{cases} \tag{2}$$

The square of a volume element $\omega = \mathbf{e}_{12\ldots n}$ $(n = p+q)$ plays an important role in the theory of Clifford algebras,

$$\omega^2 = \begin{cases} -1 & \text{if } p-q \equiv 2,3,6,7 \pmod 8, \\ +1 & \text{if } p-q \equiv 0,1,4,5 \pmod 8. \end{cases} \tag{3}$$

A center $\mathbf{Z}_{p,q}$ of the algebra $C\ell_{p,q}$ consists of the unit $\mathbf{e}_0$ and the volume element ω. The element $\omega = \mathbf{e}_{12\ldots n}$ belongs to a center when n is odd. Indeed,

$$\mathbf{e}_{12\cdots n}\mathbf{e}_i = (-1)^{n-i}\sigma(q-i)\mathbf{e}_{12\cdots i-1 i+1\cdots n},$$

$$\mathbf{e}_i\mathbf{e}_{12\cdots n} = (-1)^{i-1}\sigma(q-i)\mathbf{e}_{12\cdots i-1 i+1\cdots n},$$

therefore, $\omega \in \mathbf{Z}_{p,q}$ if and only if $n - i \equiv i - 1 \pmod 2$, that is, n is odd. Further, using (3) we obtain

$$\mathbf{Z}_{p,q} = \begin{cases} 1 & \text{if } p - q \equiv 0, 2, 4, 6 \pmod 8, \\ 1, \omega & \text{if } p - q \equiv 1, 3, 5, 7 \pmod 8). \end{cases} \tag{4}$$

In Clifford algebra $C\ell$ there exist for fundamental automorphisms.

(1) *Identity*: An automorphism $\mathcal{A} \rightarrow \mathcal{A}$ and $\mathbf{e}_i \rightarrow \mathbf{e}_i$.
This automorphism is an identical automorphism of the algebra $C\ell$. $\mathcal{A}$ is an arbitrary element of $C\ell$.

(2) *Involution*: An automorphism $\mathcal{A} \rightarrow \mathcal{A}^\star$ and $\mathbf{e}_i \rightarrow -\mathbf{e}_i$.
In more details, for an arbitrary element $\mathcal{A} \in C\ell$ there exists a decomposition $\mathcal{A} = \mathcal{A}' + \mathcal{A}''$, where $\mathcal{A}'$ is an element consisting of homogeneous odd elements, and $\mathcal{A}''$ is an element consisting of homogeneous even elements, respectively. Then the automorphism $\mathcal{A} \rightarrow \mathcal{A}^\star$ is such that the element $\mathcal{A}''$ is not changed, and the element $\mathcal{A}'$ changes sign: $\mathcal{A}^\star = -\mathcal{A}' + \mathcal{A}''$. If $\mathcal{A}$ is a homogeneous element, then

$$\mathcal{A}^\star = (-1)^k \mathcal{A}, \tag{5}$$

where k is a degree of the element. It is easy to see that the automorphism $\mathcal{A} \rightarrow \mathcal{A}^\star$ may be expressed via the volume element $\omega = \mathbf{e}_{12\cdots p+q}$:

$$\mathcal{A}^\star = \omega \mathcal{A} \omega^{-1}, \tag{6}$$

where $\omega^{-1} = (-1)^{\frac{(p+q)(p+q-1)}{2}} \omega$. When k is odd, for the basis elements $\mathbf{e}_{i_1 i_2 \cdots i_k}$ the sign changes, and when k is even, the sign is not changed.

(3) *Reversion*: An antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ and $\mathbf{e}_i \rightarrow \mathbf{e}_i$.
The antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ is a reversion of the element $\mathcal{A}$, that is the substitution of each basis element $\mathbf{e}_{i_1 i_2 \cdots i_k} \in \mathcal{A}$ by the element $\mathbf{e}_{i_k i_{k-1} \cdots i_1}$:

$$\mathbf{e}_{i_k i_{k-1} \cdots i_1} = (-1)^{\frac{k(k-1)}{2}} \mathbf{e}_{i_1 i_2 \cdots i_k}.$$

Therefore, for any $\mathcal{A} \in C\ell_{p,q}$, we have

$$\widetilde{\mathcal{A}} = (-1)^{\frac{k(k-1)}{2}} \mathcal{A}. \tag{7}$$

(4) *Conjugation*: An antiautomorpism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}^\star}$ and $\mathbf{e}_i \rightarrow -\mathbf{e}_i$.
This antiautomorphism is a composition of the antiautomorphism $\mathcal{A} \rightarrow \widetilde{\mathcal{A}}$ with the automorphism $\mathcal{A} \rightarrow \mathcal{A}^\star$. In the case of a homogeneous element, from the formulae (5) and (7), it follows

$$\widetilde{\mathcal{A}^\star} = (-1)^{\frac{k(k+1)}{2}} \mathcal{A}. \tag{8}$$

The Lipschitz group $\Gamma_{p,q}$, also called the Clifford group, introduced by Lipschitz (1886) may be defined as the subgroup of invertible elements s of the algebra $C\ell_{p,q}$:

$$\Gamma_{p,q} = \left\{ s \in C\ell^{+}_{p,q} \cup C\ell^{-}_{p,q} \mid \forall x \in \mathbb{R}^{p,q}, s\mathbf{x}s^{-1} \in \mathbb{R}^{p,q} \right\}.$$

The set $\Gamma^{+}_{p,q} = \Gamma_{p,q} \cap C\ell^{+}_{p,q}$ is called the *special Lipschitz group* (Chevalley, 1955).

Let $N : C\ell_{p,q} \to C\ell_{p,q}$, $N(\mathbf{x}) = \mathbf{x}\widetilde{\mathbf{x}}$. If $\mathbf{x} \in \mathbb{R}^{p,q}$, then $N(\mathbf{x}) = \mathbf{x}(-\mathbf{x}) = -\mathbf{x}^2 = -Q(\mathbf{x})$.
Further, the group $\Gamma_{p,q}$ has a subgroup

$$\mathbf{Pin}(p,q) = \{s \in \Gamma_{p,q} \mid N(s) = \pm 1\}. \tag{9}$$

Analogously, *a spinor group* $\mathbf{Spin}(p,q)$ is defined by the set

$$\mathbf{Spin}(p,q) = \left\{ s \in \Gamma^{+}_{p,q} \mid N(s) = \pm 1 \right\}. \tag{10}$$

It is obvious that $\mathbf{Spin}(p,q) = \mathbf{Pin}(p,q) \cap C\ell^{+}_{p,q}$. The group $\mathbf{Spin}(p,q)$ contains a subgroup

$$\mathbf{Spin}_{+}(p,q) = \{s \in \mathbf{Spin}(p,q) \mid N(s) = 1\}. \tag{11}$$

It is easy to see that the groups $O(p,q)$, $SO(p,q)$, and $SO_{+}(p,q)$ are isomorphic, respectively, to the following quotient groups

$$O(p,q) \simeq \mathbf{Pin}(p,q)/\mathbb{Z}_2, \qquad SO(p,q) \simeq \mathbf{Spin}(p,q)/\mathbb{Z}_2,$$

$$SO_{+}(p,q) \simeq \mathbf{Spin}_{+}(p,q)/\mathbb{Z}_2,$$

where the kernel $\mathbb{Z}_2 = \{1, -1\}$. Thus, the groups $\mathbf{Pin}(p,q)$, $\mathbf{Spin}(p,q)$, and $\mathbf{Spin}_{+}(p,q)$ are the double coverings of the groups $O(p,q)$, SO, and $SO_{+}(p,q)$, respectively.

On the other hand, there exists a more detailed version of the **Pin** group (9) proposed by Dąbrowski in 1988. In general, there are eight double coverings of the orthogonal group $O(p,q)$ (Blau and Dąbrowski, 1989; Dąbrowski, 1988):

$$\rho^{a,b,c} : \mathbf{Pin}^{a,b,c}(p,q) \to O(p,q),$$

where $a, b, c \in \{+, -\}$. As known, the group $O(p,q)$ consists of four connected components: identity-connected component $O_0(p,q)$, and three components corresponding to parity reversal P, time reversal T, and the combination of these two PT, that is, $O(p,q) = (O_0(p,q)) \cup P(Q_0(p,q)) \cup T(Q_0(p,q)) \cup PT(O_0(p,q))$. Further, since the four-element group (reflection group) $\{1, P, T, PT\}$ is isomorphic to the finite group $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ (Gauss–Klein veergruppe, Salingaros, 1981, 1984), then $O(p,q)$ may be represented by a semidirect product $O(p,q) \simeq O_0(p,q) \odot (\mathbb{Z}_2 \otimes \mathbb{Z}_2)$. The signs of a, b, and c correspond to the signs of the squares of the elements in $\mathbf{Pin}^{a,b,c}(p,q)$ that cover space reflection P, time reversal T, and a combination of these two PT ($a = -P^2, b = T^2, c = -(PT)^2$) in Dąbrowski's (1988) notation and $a = P^2$, $b = T^2$, $c = (PT)^2$ in Chamblin's (1994) notation, which we will use below). An explicit form of the group $\mathbf{Pin}^{a,b,c}(p,q)$ is given by the following semidirect product

$$\mathbf{Pin}^{a,b,c}(p,q) \simeq \frac{(\mathbf{Spin}_0(p,q) \odot C^{a,b,c})}{\mathbb{Z}_2}, \tag{12}$$

where $C^{a,b,c}$ are the four double coverings of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$. All the eight double coverings of the orthogonal group $O(p,q)$ are given in the following table:

$a\ b\ c$	$C^{a,b,c}$	Remark
$+\ +\ +$	$\mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$	$PT = TP$
$+\ -\ -$	$\mathbb{Z}_2 \otimes \mathbb{Z}_4$	$PT = TP$
$-\ +\ -$	$\mathbb{Z}_2 \otimes \mathbb{Z}_4$	$PT = TP$
$-\ -\ +$	$\mathbb{Z}_2 \otimes \mathbb{Z}_4$	$PT = TP$
$-\ -\ -$	Q_4	$PT = -TP$
$-\ +\ +$	D_4	$PT = -TP$
$+\ -\ +$	D_4	$PT = -TP$
$+\ +\ -$	D_4	$PT = -TP$

Here $\mathbb{Z}_4$, Q_4, and D_4 are complex, quaternion, and dihedral groups, respectively. According to Dąbrowski (1988) the group $\mathbf{Pin}^{a,b,c}(p,q)$ satisfying the condition $PT = -TP$ is called *Cliffordian*, and *non-Cliffordian* when $PT = TP$.

One of the most fundamental theorems in the theory of associative algebras is as follows:

Theorem 1 *(Wedderburn–Artin). Any finite-dimensional associative simple algebra $\mathfrak{A}$ over the field $\mathbb{F}$ is isomorphic to a full matrix algebra $\mathsf{M}_n(\mathbb{K})$, where n is a natural number defined unambiguously, and $\mathbb{K}$ a division ring defined with an accuracy of isomorphism.*

In accordance with this theorem, all properties of the initial algebra $\mathfrak{A}$ are isomorphically transferred to the matrix algebra $\mathsf{M}_n(\mathbb{K})$. Later on we will widely use this theorem. In its turn, for the Clifford algebra $C\ell_{p,q}$ over the field $\mathbb{F} = \mathbb{R}$ we have an isomorphism $C\ell_{p,q} \simeq \mathrm{End}_{\mathbb{K}}(I_{p,q}) \simeq \mathsf{M}_{2^m}(\mathbb{K})$, where $m = \frac{p+q}{2}$, $I_{p,q} = C\ell_{p,q} f$ is a minimal left ideal of $C\ell_{p,q}$, and $\mathbb{K} = f C\ell_{p,q} f$ is a division ring of $C\ell_{p,q}$. The primitive idempotent of the algebra $C\ell_{p,q}$ has a form

$$f = \frac{1}{2}(1 \pm \mathbf{e}_{\alpha_1})\frac{1}{2}(1 \pm \mathbf{e}_{\alpha_2}) \cdots \frac{1}{2}(1 \pm \mathbf{e}_{\alpha_k}),$$

where $\mathbf{e}_{\alpha_1}, \mathbf{e}_{\alpha_2}, \ldots, \mathbf{e}_{\alpha_k}$ are commuting elements with square 1 of the canonical basis of $C\ell_{p,q}$ generating a group of order 2^k. The values of k are defined by a formula $k = q - r_{q-p}$, where r_i are the Radon–Hurwitz numbers (Hurwitz, 1923; Radon, 1922), values of which form a cycle of period 8: $r_{i+8} = r_i + 4$. The values of all r_i are

$$\begin{array}{c|cccccccc} i & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline r_i & 0 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \end{array}.$$

All the Clifford algebras $C\ell_{p,q}$ over the field $\mathbb{F} = \mathbb{R}$ are divided into eight different types with the following division ring structure:

- I. Central simple algebras.
  - (1) Two types $p - q \equiv 0, 2 \pmod 8$, with a division ring $\mathbb{K} \simeq \mathbb{R}$.
  - (2) Two types $p - q \equiv 3, 7 \pmod 8$, with a division ring $\mathbb{K} \simeq \mathbb{C}$.
  - (3) Two types $p - q \equiv 4, 6 \pmod 8$, with a division ring $\mathbb{K} \simeq \mathbb{H}$.
- II. Semisimple algebras.
  - (4) The type $p - q \equiv 1 \pmod 8$, with a double division ring $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$.
  - (5) The type $p - q \equiv 5 \pmod 8$, with a double quaternionic division ring $\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}$.

Over the field $\mathbb{F} = \mathbb{C}$ there is an isomorphism $\mathbb{C}_n \simeq \mathsf{M}_{2^{n/2}}(\mathbb{C})$ and there are two different types of complex Clifford algebras $\mathbb{C}_n$: $n \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$.

In virtue of the Wedderburn–Artin theorem, all fundamental automorphisms of $C\ell$ are transferred to the matrix algebra. Matrix representations of the fundamental automorphisms of $\mathbb{C}_n$ was first obtained by Rashevskii in 1955 (Rashevskii, 1955): (1) Involution: $\mathsf{A}^\star = \mathsf{W}\mathsf{A}\mathsf{W}^{-1}$, where W is a matrix of the automorphism $\star$ (matrix representation of the volume element ω); (2) Reversion: $\widetilde{\mathsf{A}} = \mathsf{E}\mathsf{A}^\top\mathsf{E}^{-1}$, where E is a matrix of the antiautomorphism $\widetilde{}$ satisfying the conditions $\mathcal{E}_i\mathsf{E} - \mathsf{E}\mathcal{E}_i^\top = 0$ and $\mathsf{E}^\top = (-1)^{\frac{m(m-1)}{2}}\mathsf{E}$ ($\mathcal{E}_i = \gamma(\mathbf{e}_i)$ are matrix representations of the units of the algebra $C\ell$); (3) Conjugation: $\widetilde{\mathsf{A}^\star} = \mathsf{C}\mathsf{A}^\top\mathsf{C}^{-1}$, where $\mathsf{C} = \mathsf{E}\mathsf{W}^\top$ is a matrix of the antiautomorphism $\widetilde{\star}$ satisfying the conditions $\mathsf{C}\mathcal{E}^\top + \mathcal{E}_i\mathsf{C} = 0$ and $\mathsf{C}^\top = (-1)^{\frac{m(m+1)}{2}}\mathsf{C}$.

In the recent paper Varlamov (1999), it has been shown that space reversal P, time reversal T, and combination PT correspond to the fundamental automorphisms $\mathcal{A} \to \mathcal{A}^\star$, $\mathcal{A} \to \widetilde{\mathcal{A}}$, and $\mathcal{A} \to \widetilde{\mathcal{A}^\star}$ respectively. Moreover, there is an isomorphism between the discrete subgroup $\{1, P, T, PT\} \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ ($P^2 = T^2 = (PT)^2 = 1, PT = TP$) of $O(p, q)$ and an automorphism group $\mathrm{Aut}(C\ell) = \{\mathrm{Id}, \star, \widetilde{}, \widetilde{\star}\}$:

	Id	$\star$	$\sim$	$\widetilde{\star}$
Id	Id	$\star$	$\sim$	$\widetilde{\star}$
$\star$	$\star$	Id	$\widetilde{\star}$	$\sim$
$\sim$	$\sim$	$\widetilde{\star}$	Id	$\star$
$\widetilde{\star}$	$\widetilde{\star}$	$\sim$	$\star$	Id

$\sim$

	1	P	T	PT
1	1	P	T	PT
P	P	1	PT	T
T	T	PT	1	P
PT	PT	T	P	1

Further, in the case $P^2 = T^2 = (PT)^2 = \pm 1$ and $PT = -TP$, there is an isomorphism between the group $\{1, P, T, PT\}$ and an automorphism group $\mathsf{Aut}(C\ell) = \{\mathsf{I}, \mathsf{W}, \mathsf{E}, \mathsf{C}\}$. So, for the Dirac algebra $\mathbb{C}_4$ in the canonical γ-basis, there exists a standard (Wigner) representation $P = \gamma_0$ and $T = \gamma_1\gamma_3$ (Berestetskii *et al.*, 1982), therefore, $\{1, P, T, PT\} = \{1, \gamma_0, \gamma_1\gamma_3, \gamma_0\gamma_1\gamma_3\}$. On the other hand, in the γ-basis, an automorphism group of $\mathbb{C}_4$ has a form $\mathsf{Aut}(\mathbb{C}_4) = \{\mathsf{I}, \mathsf{W}, \mathsf{E}, \mathsf{C}\} = \{\mathsf{I}, \gamma_0\gamma_1\gamma_2\gamma_3, \gamma_1\gamma_3, \gamma_0\gamma_2\}$. It has been shown (Varlamov, 1999) that $\{1, P, T, PT\} = \{1, \gamma_0, \gamma_1\gamma_3, \gamma_0\gamma_1\gamma_3\} \simeq \mathsf{Aut}(\mathbb{C}_4) \simeq \mathbb{Z}_4$, where $\mathbb{Z}_4$ is a complex group with the signature $(+, -, -)$. Generalizations of these results onto the algebras $\mathbb{C}_n$ are contained in the following two theorems:

Theorem 2 *(Varlamov, 1999). Let* $\mathsf{Aut}(\mathbb{C}_n) = \{\mathsf{I}, \mathsf{W}, \mathsf{E}, \mathsf{C}\}$ *be a group of the fundamental automorphisms of the algebra* $\mathbb{C}_n$ $(n = 2m)$*, where* $\mathsf{W} = \mathcal{E}_1\mathcal{E}_2 \cdots \mathcal{E}_m\mathcal{E}_{m+1} \mathcal{E}_{m+2} \cdots \mathcal{E}_{2m}$, $\mathsf{E} = \mathcal{E}_1\mathcal{E}_2 \cdots \mathcal{E}_m$*, and* $\mathsf{C} = \mathcal{E}_{m+1}\mathcal{E}_{m+2} \cdots \mathcal{E}_{2m}$ *if* $m \equiv 1 \pmod 2$*, and* $\mathsf{E} = \mathcal{E}_{m+1}\mathcal{E}_{m+2} \cdots \mathcal{E}_{2m}$, $\mathsf{C} = \mathcal{E}_1 E_2 \cdots \mathcal{E}_m$ *if* $m \equiv 0 \pmod 2$*. Let* $\mathsf{Aut}_-(\mathbb{C}_n)$ *and* $\mathsf{Aut}_+(\mathbb{C}_n)$ *be the automorphism groups, in which all the elements commute* ($m \equiv 0 \pmod 2$) *and anticommute* ($m \equiv 1 \pmod 2$)*, respectively. Then over the field* $\mathbb{F} = \mathbb{C}$*, there exist only two non-isomorphic groups:* $\mathsf{Aut}_-(\mathbb{C}_n) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ *with the signature* $(+, +, +)$ *if* $n \equiv 0, 4 \pmod 8$ *and* $\mathsf{Aut}_+(\mathbb{C}_n) \simeq Q_4/\mathbb{Z}_2$ *with the signature* $(-, -, -)$ *if* $n \equiv 2, 6 \pmod 8$.

Theorem 3 *(Varlamov, 1999). Let* $\mathbf{Pin}^{a,b,c}(n, \mathbb{C})$ *be a double covering of the complex orthogonal group* $O(n, \mathbb{C})$ *of the space* $\mathbb{C}^n$ *associated with the complex algebra* $\mathbb{C}_n$*. A dimensionality of the algebra* $\mathbb{C}_n$ *is even* ($n = 2m$)*, squares of the symbols* $a, b, c \in \{-, +\}$ *correspond to squares of the elements of the finite group* $\mathsf{Aut} = \{\mathsf{I}, \mathsf{W}, \mathsf{E}, \mathsf{C}\}$: $a = \mathsf{W}^2$, $b = \mathsf{E}^2$, $c = \mathsf{C}^2$*, where* W, E*, and* C *are the matrices of the fundamental automorphisms* $\mathcal{A} \to \mathcal{A}^\star$, $\mathcal{A} \to \widetilde{\mathcal{A}}$*, and* $\mathcal{A} \to \widetilde{\mathcal{A}^\star}$ *of* $\mathbb{C}_n$*, respectively. Then over the field* $\mathbb{F} = \mathbb{C}$*, for the algebra* $\mathbb{C}_n$ *there are two non-isomorphic double coverings of the group* $O(n, \mathbb{C})$*:*

(1) *A non-Cliffordian group*

$$\mathbf{Pin}^{+,+,+}(n, \mathbb{C}) \simeq \frac{(\mathbf{Spin}_0(n, \mathbb{C}) \odot \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2)}{\mathbb{Z}_2},$$

if $n \equiv 0, 4 \pmod 8$.

(2) *A Cliffordian group*

$$\mathbf{Pin}^{-,-,-}(n, \mathbb{C}) \simeq \frac{(\mathbf{Spin}_0(n, \mathbb{C}) \odot Q_4)}{\mathbb{Z}_2},$$

if $n \equiv 2, 6 \pmod 8$.

3. DISCRETE SYMMETRIES OVER THE FIELD $\mathbb{F}=\mathbb{R}$

Theorem 4. *Let $C\ell_{p,q}$ be a Clifford algebra over a field $\mathbb{F}=\mathbb{R}$ and let $\mathsf{Aut}(C\ell_{p,q})=\{\mathsf{I},\mathsf{W},\mathsf{E},\mathsf{C}\}$ be a group of fundamental automorphisms of the algebra $C\ell_{p,q}$. Then for eight types of the algebras $C\ell_{p,q}$ there exist, depending upon a division ring structure of $C\ell_{p,q}$, following isomorphisms between finite groups and groups $\mathsf{Aut}(C\ell_{p,q})$ with different signatures (a,b,c), where $a,b,c\in\{-,+\}$:*

(1) $\mathbb{K}\simeq\mathbb{R}$, *types* $p-q\equiv 0,2 \pmod 8$.
If $\mathsf{E}=\mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q}$ and $\mathsf{C}=\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p$, then Abelian groups $\mathsf{Aut}_-(C\ell_{p,q})\simeq\mathbb{Z}_2\otimes\mathbb{Z}_2$ with the signature $(+,+,+)$ and $\mathsf{Aut}_-(C\ell_{p,q})\simeq\mathbb{Z}_4$ with the signature $(+,-,-)$ exist at $p,q\equiv 0 \pmod 4$ and $p,q\equiv 2 \pmod 4$, respectively, for the type $p-q\equiv 0 \pmod 8$, and also Abelian groups $\mathsf{Aut}_-(C\ell_{p,q})\simeq\mathbb{Z}_4$ with the signature $(-,-,+)$ and $\mathsf{Aut}_-(C\ell_{p,q})\simeq\mathbb{Z}_4$ with the signature $(-,+,-)$ exist at $p\equiv 0 \pmod 4$, $q\equiv 2 \pmod 4$ and $p\equiv 2 \pmod 4$, $q\equiv 0 \pmod 4$ for the type $p-q\equiv 2 \pmod 8$, respectively.

If $\mathsf{E}=\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p$ and $\mathsf{C}=\mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q}$, then non-Abelian groups $\mathsf{Aut}_+(C\ell_{p,q})\simeq D_4/\mathbb{Z}_2$ with the signature $(+,-,+)$ and $\mathsf{Aut}_+(C\ell_{p,q})\simeq D_4/\mathbb{Z}_2$ with the signature $(+,+,-)$ exist at $p,q\equiv 3 \pmod 4$ and $p,q\equiv 1 \pmod 4$, respectively, for the type $p-q\equiv 0 \pmod 8$, and also non-Abelian groups $\mathsf{Aut}_+(C\ell_{p,q})\simeq Q_4/\mathbb{Z}_2$ with $(-,-,-)$ and $\mathsf{Aut}_+(C\ell_{p,q})\simeq D_4/\mathbb{Z}_2$ with $(-,+,+)$ exist at $p\equiv 3 \pmod 4$, $q\equiv 1 \pmod 4$ and $p\equiv 1 \pmod 4$, $q\equiv 3 \pmod 4$ for the type $p-q\equiv 2 \pmod 8$, respectively.

(2) $\mathbb{K}\simeq\mathbb{H}$, *types* $p-q\equiv 4,6 \pmod 8$.
If $\mathsf{E}=\mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}$ is a product of k skewsymmetric matrices (among which l matrices have a square $+\mathsf{I}$ and t matrices have a square $-\mathsf{I}$) and $\mathsf{C}=\mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}$ is a product of $p+q-k$ symmetric matrices (among which h matrices have a square $+\mathsf{I}$ and g have a square $-\mathsf{I}$), then at $k\equiv 0 \pmod 2$ for the type $p-q\equiv 4 \pmod 8$ there exist Abelian groups $\mathsf{Aut}_-(C\ell_{p,q})\simeq\mathbb{Z}_2\otimes\mathbb{Z}_2$ with $(+,+,+)$ and $\mathsf{Aut}_-(C\ell_{p,q})\simeq\mathbb{Z}_4$ with $(+,-,-)$ if $l-t$, $h-g\equiv 0,1,4,5 \pmod 8$ and $l-t$, $h-g\equiv 2,3,6,7 \pmod 8$, respectively. And also at $k\equiv 0 \pmod 2$ for the type $p-q\equiv 6 \pmod 8$ there exist $\mathsf{Aut}_-(C\ell_{p,q})\simeq\mathbb{Z}_4$ with $(-,+,-)$ and $\mathsf{Aut}_-(C\ell_{p,q})\simeq\mathbb{Z}_4$ with $(-,-,+)$ if $l-t\equiv 0,1,4,5 \pmod 8$, $h-g\equiv 2,3,6,7 \pmod 8$ and $l-t\equiv 2,3,6,7 \pmod 8$, $h-g\equiv 0,1,4,5 \pmod 8$, respectively.

Inversely, if $\mathsf{E}=\mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}$ is a product of $p+q-k$ symmetric matrices and $\mathsf{C}=\mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}$ is a product of k skewsymmetric matrices, then at $k\equiv 1 \pmod 2$ for the type $p-q\equiv 4 \pmod 8$ there exist non-Abelian groups $\mathsf{Aut}_+(C\ell_{p,q})\simeq D_4/\mathbb{Z}_2$ with $(+,-,+)$

and $\mathsf{Aut}_+(C\ell_{p,q}) \simeq D_4/\mathbb{Z}_2$ *with* $(+,+,-)$ *if* $h-g \equiv 2,3,6,7 \pmod 8$, $l-t \equiv 0,1,4,5 \pmod 8$ *and* $h-g \equiv 0,1,4,5 \pmod 8$, $l-t \equiv 2,3,6,7 \pmod 8$, *respectively. And also at* $k \equiv 1 \pmod 2$ *for the type* $p-q \equiv 6 \pmod 8$ *there exist* $\mathsf{Aut}_+(C\ell_{p,q}) \simeq Q_4/\mathbb{Z}_2$ *with* $(-,-,-)$ *and* $\mathsf{Aut}_+(C\ell_{p,q}) \simeq D_4/\mathbb{Z}_2$ *with* $(-,+,+)$ *if* $h-g, l-t \equiv 2,3,6,7 \pmod 8$ *and* $h-g, l-t \equiv 0,1,4,5 \pmod 8$, *respectively.*

(3) $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$, $\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}$, *types* $p-q \equiv 1,5 \pmod 8$.
For the algebras $C\ell_{0,q}$ *of the types* $p-q \equiv 1,5 \pmod 8$ *there exist Abelian automorphism groups with the signatures* $(-,-,+)$, $(-,+,-)$ *and non-Abelian automorphism groups with the signatures* $(-,-,-)$, $(-,+,+)$. *Correspondingly, for the algebras* $C\ell_{p,0}$ *of the types* $p-q \equiv 1,5 \pmod 8$ *there exist Abelian groups with* $(+,+,+)$, $(+,-,-)$ *and non-Abelian groups with* $(+,-,+)$, $(+,+,-)$. *In a general case for* $C\ell_{p,q}$, *the types* $p-q \equiv 1,5 \pmod 8$ *admit all eight automorphism groups.*

(4) $\mathbb{K} = \mathbb{C}$, *types* $p-q \equiv 3,7 \pmod 8$.
The types $p-q \equiv 3,7 \pmod 8$ *admit the Abelian group* $\mathsf{Aut}_-(C\ell_{p,q}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ *with the signature* $(+,+,+)$ *if* $p \equiv 0 \pmod 2$ *and* $q \equiv 1 \pmod 2$, *and also non-Abelian group* $\mathsf{Aut}_+(C\ell_{p,q}) \simeq Q_4/\mathbb{Z}_2$ *with the signature* $(-,-,-)$ *if* $p \equiv 1 \pmod 2$ *and* $q \equiv 0 \pmod 2$.

Proof: Before we proceed to prove this theorem, let us consider in more details a matrix (spinor) representation of the antiautomorphisms $\mathcal{A} \to \widetilde{\mathcal{A}}$ and $\mathcal{A} \to \widetilde{\mathcal{A}^\star}$. According to Wedderburn-Artin theorem, the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$ corresponds to an antiautomorphism of the full matrix algebra $\mathsf{M}_{2^m}(\mathbb{K})$: $\mathsf{A} \to \mathsf{A}^\top$, in virtue of the well-known relation $(\mathsf{AB})^\top = \mathsf{B}^\top\mathsf{A}^\top$, where T is a symbol of transposition. On the other hand, in the matrix representation of the elements $\mathcal{A} \in C\ell_{p,q}$, for the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$ we have $\mathsf{A} \to \widetilde{\mathsf{A}}$. A composition of the two antiautomorphisms, $\mathsf{A}^\top \to \mathsf{A} \to \widetilde{\mathsf{A}}$, gives an automorphism $\mathsf{A}^\top \to \widetilde{\mathsf{A}}$, which is an internal automorphism of the algebra $\mathsf{M}_{2^m}(\mathbb{K})$:

$$\widetilde{\mathsf{A}} = \mathsf{E}\mathsf{A}^\top\mathsf{E}^{-1}, \tag{13}$$

where E is a matrix, by means of which the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$ is expressed in the matrix representation of the algebra $C\ell_{p,q}$. Under action of the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$ the units of $C\ell_{p,q}$ remain unaltered, $\mathbf{e}_i \to \mathbf{e}_i$; therefore in the matrix representation, we must demand $\mathcal{E}_i \to \mathcal{E}_i$, where $\mathcal{E}_i = \gamma(\mathbf{e}_i)$ also. Therefore, for the definition of the matrix E in accordance with (13), we have

$$\mathcal{E}_i \to \mathcal{E}_i = \mathsf{E}\mathcal{E}^\top\mathsf{E}^{-1}. \tag{14}$$

Or, let $\{\mathcal{E}_{\alpha_i}\}$ be a set consisting of symmetric matrices ($\mathcal{E}^\top_{\alpha_i} = \mathcal{E}_{\alpha_i}$) and let $\{\mathcal{E}_{\beta_j}\}$ be a set consisting of skewsymmetric matrices ($\mathcal{E}^\top_{\beta_j} = -\mathcal{E}_{\beta_j}$). Then the transformation

(14) may be rewritten in the following form:

$$\mathcal{E}_{\alpha_i} \to \mathcal{E}_{\alpha_i} = \mathsf{E}\mathcal{E}_{\alpha_i}\mathsf{E}^{-1}, \qquad \mathcal{E}_{\beta_j} \to \mathcal{E}_{\beta_j} = -\mathsf{E}\mathcal{E}_{\beta_j}\mathsf{E}^{-1}.$$

Whence

$$\mathcal{E}_{\alpha_i}\mathsf{E} = \mathsf{E}\mathcal{E}_{\alpha_i}, \qquad \mathcal{E}_{\beta_j}\mathsf{E} = -\mathsf{E}\mathcal{E}_{\beta_j}. \tag{15}$$

Thus, the matrix E of the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$ commutes with a symmetric part of the spinbasis of the algebra $C\ell_{p,q}$ and anticommutes with a skewsymmetric part. An explicit form of the matrix E in dependence on the type of the algebras $C\ell_{p,q}$ will be found later, but first let us define a general form of E, that is, let us show that for the form of E there are only two possibilities: (1) E is a product of symmetric matrices or (2) E is a product of skewsymmetric matrices. Let us prove this assertion another way: Let $\mathsf{E} = \mathcal{E}_{\alpha_1}\mathcal{E}_{\alpha_2}\cdots\mathcal{E}_{\alpha_s}\mathcal{E}_{\beta_1}\mathcal{E}_{\beta_2}\cdots\mathcal{E}_{\beta_k}$ be a product of s symmetric and k skewsymmetric matrices. At this point, $1 < s + k \leq p + q$. The permutation condition of the matrix E with the symmetric basis matrices $\mathcal{E}_{\alpha_i}$ have a form

$$\begin{aligned} \mathcal{E}_{\alpha_i}\mathsf{E} &= (-1)^{i-1}\sigma(\alpha_i)\mathcal{E}_{\alpha_1}\cdots\mathcal{E}_{\alpha_{i-1}}\mathcal{E}_{\alpha_{i+1}}\cdots\mathcal{E}_{\alpha_s}\mathcal{E}_{\beta_1}\cdots\mathcal{E}_{\beta_k}, \\ \mathsf{E}\mathcal{E}_{\alpha_i} &= (-1)^{k+s-i}\sigma(\alpha_i)\mathcal{E}_{\alpha_1}\cdots\mathcal{E}_{\alpha_{i-1}}\mathcal{E}_{\alpha_{i+1}}\cdots\mathcal{E}_{\alpha_s}\mathcal{E}_{\beta_1}\cdots\mathcal{E}_{\beta_k}. \end{aligned} \tag{16}$$

From here we obtain a comparison $k + s - i \equiv i - 1 \pmod 2$, that is, at $k + s \equiv 0 \pmod 2$, E and $\mathcal{E}_{\alpha_i}$ anticommute and at $k + s \equiv 1 \pmod 2$ commute. Analogously, for the skewsymmetric part we have

$$\begin{aligned} \mathcal{E}_{\beta_j}\mathsf{E} &= (-1)^{s+j-1}\sigma(\beta_j)\mathcal{E}_{\alpha_1}\cdots\mathcal{E}_{\alpha_s}\mathcal{E}_{\beta_1}\cdots\mathcal{E}_{\beta_{j-1}}\mathcal{E}_{\beta_{j+1}}\cdots\mathcal{E}_{\beta_k}, \\ \mathsf{E}\mathcal{E}_{\beta_j} &= (-1)^{k-j}\sigma(\beta_i)\mathcal{E}_{\alpha_1}\cdots\mathcal{E}_{\alpha_s}\mathcal{E}_{\beta_1}\cdots\mathcal{E}_{\beta_{j-1}}\mathcal{E}_{\beta_{j+1}}\cdots\mathcal{E}_{\beta_k}. \end{aligned} \tag{17}$$

From the comparison $k - s \equiv 2j - 1 \pmod 2$, it follows that at $k - s \equiv 0 \pmod 2$, E and $\mathcal{E}_{\beta_j}$ anticommute and at $k - s \equiv 1 \pmod 2$ commute. Let $k + s = p + q$, then from (16) we see that at $p + q \equiv \pmod 2$, E and $\mathcal{E}_{\alpha_i}$ anticommute, which is inconsistent with (15). The case $p + q \equiv 1 \pmod 2$ is excluded, since a dimensionality of $C\ell_{p,q}$ is even (in the case of odd dimensionality the algebra $C\ell_{p+1,q}$ ($C\ell_{p,q+1}$) is isomorphic to $\mathrm{End}_{\mathbb{K}\oplus\hat{\mathbb{K}}}(I_{p,q} \oplus \hat{I}_{p,q}) \simeq \mathsf{M}_{2^m}(\mathbb{K}) \oplus \mathsf{M}_{2^m}(\mathbb{K})$, where $m = (p + q)/2$. Let suppose now that $k + s < p + q$, that is, let us eliminate from the product E one symmetric matrix, then $k + s \equiv 1 \pmod 2$ and in virtue of (16) the matrices $\mathcal{E}_{\alpha_i}$ that belong to E commute with E, but the matrix that does not belong to E anticommute with E. Thus, we came to a contradiction with (15). It is obvious that elimination of two, three, or more symmetric matrices from E gives an analogous situation. Now, let us eliminate from E one skewsymmetric matrix, then $k + s \equiv 1 \pmod 2$ and in virtue of (16) E and all $\mathcal{E}_{\alpha_i}$ commute with each other. Further, in virtue of (17) the matrices $\mathcal{E}_{\beta_j}$ that belong to the product E commute with E, whereas the the skewsymmetric matrix that does not belong to

E anticommute with E. Therefore we again come to a contradiction with (15). We come to an analogous situation if we eliminate two, three, or more skewsymmetric matrices. Thus, the product E does not contain simultaneously symmetric and skewsymmetric matrices. Hence it follows that the matrix of the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$ is a product of only symmetric or only skewsymmetric matrices.

Further, the matrix representations of the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}^\star}$: $\widetilde{\mathsf{A}}^\star = \mathsf{C}\mathsf{A}^\top\mathsf{C}^{-1}$ is defined in a similar manner. First of all, since under action of the antiautomorphism $\widetilde{\star}$ we have $\mathbf{e}_i \to -\mathbf{e}_i$, in the matrix representation we must demand $\mathcal{E}_i \to -\mathcal{E}_i$ also, or

$$\mathcal{E}_i \to -\mathcal{E}_i = \mathsf{C}\mathcal{E}^\top\mathsf{C}^{-1}. \tag{18}$$

Taking into account the symmetric $\{\mathcal{E}_{\alpha_i}\}$ and the skewsymmetric $\{\mathcal{E}_{\beta_j}\}$ parts of the spinbasis, we can write the transformation (18) in the form

$$\mathcal{E}_{\alpha_i} \to -\mathcal{E}_{\alpha_i} = \mathsf{C}\mathcal{E}_{\alpha_i}\mathsf{C}^{-1}, \qquad \mathcal{E}_{\beta_j} \to \mathcal{E}_{\beta_j} = \mathsf{C}\mathcal{E}_{\beta_j}\mathsf{C}^{-1}.$$

Hence it follows

$$\mathsf{C}\mathcal{E}_{\alpha_i} = -\mathcal{E}_{\alpha_i}\mathsf{C}, \qquad \mathcal{E}_{\beta_j}\mathsf{C} = \mathsf{C}\mathcal{E}_{\beta_j}. \tag{19}$$

Thus, in contrast with (15) the matrix C of the antiautomorphism $\widetilde{\star}$ anticommutes with the symmetric part of the spinbasis of the algebra $C\ell_{p,q}$ and commutes with the skewsymmetric part of the same spinbasis. Further, in virtue of (6) a matrix representation of the automorphism $\star$ is defined as follows

$$\mathsf{A}^\star = \mathsf{W}\mathsf{A}\mathsf{W}^{-1}, \tag{20}$$

where W is a matrix representation of the volume element ω. The antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}^\star}$, in turn, is the composition of the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$ with the automorphism $\mathcal{A} \to \mathcal{A}^\star$; therefore, from (13) and (20) it follows (recall that the order of the composition of the transformations (13) and (20) is not important, since $\widetilde{\mathcal{A}^\star} = (\widetilde{\mathcal{A}})^\star = (\widetilde{\mathcal{A}^\star}))$: $\widetilde{\mathsf{A}^\star} = \mathsf{W}\mathsf{E}\mathsf{A}^\top\mathsf{E}^{-1}\mathsf{W}^{-1} = \mathsf{E}(\mathsf{W}\mathsf{A}\mathsf{W}^{-1})^\top\mathsf{E}^{-1}$, or

$$\widetilde{\mathsf{A}}^\star = (\mathsf{W}\mathsf{E})\mathsf{A}^\top(\mathsf{W}\mathsf{E})^{-1} = (\mathsf{E}\mathsf{W})\mathsf{A}^\top(\mathsf{E}\mathsf{W})^{-1}, \tag{21}$$

since $\mathsf{W}^{-1} = \mathsf{W}^\top$. Therefore, $\mathsf{C} = \mathsf{E}\mathsf{W}$ or $\mathsf{C} = \mathsf{W}\mathsf{E}$. By this reason a general form of the matrix C is similar to the form of the matrix E, that is, C is a product of symmetric or skewsymmetric matrices only.

Let us consider in sequence definitions and permutation conditions of matrices of the fundamental automorphisms (which are the elements of the groups $\mathsf{Aut}(C\ell_{p,q})$) for all eight types of the algebras $C\ell_{p,q}$, depending upon the division ring structure.

(1) The type $p - q \equiv 0 \pmod 8$, $\mathbb{K} \simeq \mathbb{R}$.

In this case according to Wedderburn–Artin theorem there is an isomorphism $C\ell_{p,q} \simeq \mathsf{M}_{2^m}(\mathbb{R})$, where $m = \frac{p+q}{2}$. First, let consider a case $p = q = m$. In the full matrix algebra $\mathsf{M}_{2^m}(\mathbb{R})$, in accordance with the signature of the algebra $C\ell_{p,q}$ a choice of the symmetric and skewsymmetric matrices $\mathcal{E}_i = \gamma(\mathbf{e}_i)$ is hardly fixed.

$$\mathcal{E}_i^\top = \begin{cases} \mathcal{E}_i, & \text{if } 1 \leq i \leq m; \\ -\mathcal{E}_i, & \text{if } m+1 \leq i \leq 2m \end{cases} \tag{22}$$

That is, at this point the matrices of the first and second half of the basis have a square $+\mathsf{I}$ and $-\mathsf{I}$, respectively. Such a form of the basis (22) is explained by the following reason: Over the field $\mathbb{R}$ there exist only symmetric matrices with a square $+\mathsf{I}$, and there exist no symmetric matrices with a square $-\mathsf{I}$. Inversely, skewsymmetric matrices over the field $\mathbb{R}$ only have a square $-\mathsf{I}$. Therefore, in this case the matrix of the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$ is a product of m symmetric matrices, $\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m$, or is a product of m skewsymmetric matrices, $\mathsf{E} = \mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{2m}$. In accordance with (15), let us find permutation conditions of the matrix E with the basis matrices $\mathcal{E}_i$. If $\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m$, and $\mathcal{E}_i$ belong to the first half of the basis (22), $1 \leq i \leq m$, then

$$\begin{aligned} \mathsf{E}\mathcal{E}_i &= (-1)^{m-i}\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_{i-1}\mathcal{E}_{i+1}\cdots\mathcal{E}_m, \\ \mathcal{E}_i\mathsf{E} &= (-1)^{i-1}\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_{i-1}\mathcal{E}_{i+1}\cdots\mathcal{E}_m. \end{aligned} \tag{23}$$

Therefore, we have a comparison $m - i \equiv i - 1 \pmod 2$, whence $m \equiv 2i - 1 \pmod 2$. Thus, the matrix E anticommutes at $m \equiv 0 \pmod 2$ and commutes at $m \equiv 1 \pmod 2$ with the basis matrices $\mathcal{E}_i$. Further, let $\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m$, and $\mathcal{E}_i$ belong to the second half of the basis $m+1 \leq i \leq 2m$, then

$$\mathsf{E}\mathcal{E}_i = (-1)^m\mathcal{E}_i\mathsf{E}. \tag{24}$$

Therefore at $m \equiv 0 \pmod 2$, E commutes and at $m \equiv 1 \pmod 2$ anticommutes with the matrices of the second half of the basis.

Let now $\mathsf{E} = \mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{2m}$ be a product of m skewsymmetric matrices, then

$$\mathsf{E}\mathcal{E}_i = (-1)^m\mathcal{E}_i\mathsf{E} \quad 1 \leq i \leq m \tag{25}$$

and

$$\begin{aligned} \mathsf{E}\mathcal{E}_i &= -(-1)^{m-i}\mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{i-1}\mathcal{E}_{i+1}\cdots\mathcal{E}_{2m}, \\ \mathcal{E}_i\mathsf{E} &= -(-1)^{i-i}\mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{i-1}\mathcal{E}_{i+1}\cdots\mathcal{E}_{2m}, \end{aligned} \quad m+1 \leq i \leq 2m \tag{26}$$

that is, at $m \equiv 0 \pmod 2$ E commutes with the matrices of the first half of the basis (22) and anticommutes with the matrices of the second half of (22). At $m \equiv 1 \pmod 2$ E anticommutes and commutes with the first and the second half of the basis (22), respectively.

Let us find permutation conditions of the matrix E with a matrix W of the volume element (a matrix of the automorphism $\star$). Let $\mathsf{E} = \mathcal{E}_i\mathcal{E}_2\cdots\mathcal{E}_m$, then

$$\begin{aligned}\mathsf{EW} &= \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_{2m} = (-1)^{\frac{m(m-1)}{2}}\mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{2m},\\ \mathsf{WE} &= \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_{2m}\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m = (-1)^{\frac{m(3m-1)}{2}}\mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{2m}.\end{aligned} \tag{27}$$

Whence $\frac{m(3m-1)}{2} \equiv \frac{m(m-1)}{2} \pmod 2$ and, therefore at $m \equiv 0 \pmod 2$, E and W commute, and at $m \equiv 1 \pmod 2$ anticommute. It is easy to verify that analogous conditions take place if $\mathsf{E} = \mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{2m}$ is the product of skewsymmetric matrices.

Since $\mathsf{C} = \mathsf{EW}$, a matrix of the antiautomorphism $\widetilde{\star}$ has a form $\mathsf{C} = \mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{2m}$ if $\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m$ and correspondingly, $\mathsf{C} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m$ if $\mathsf{E} = \mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{2m}$. Therefore, permutation conditions of the matrices C and W would be the same as that of E and W, that is, C and W commute if $m \equiv 0 \pmod 2$ and anticommute if $m \equiv 1 \pmod 2$. It is easy to see that permutation conditions of the matrix C with the basis matrices $\mathcal{E}_i$ coincide with (23)–(26).

Out of dependence on the choice of the matrices E and C, the permutation conditions between them in any of the two cases considered previously are defined by the following relation

$$\mathsf{EC} = (-1)^{m^2}\mathsf{CE}, \tag{28}$$

that is, the matrices E and C commute if $m \equiv 0 \pmod 2$ and anticommute if $m \equiv 1 \pmod 2$.

Now, let us consider squares of the elements of the automorphism groups $\mathsf{Aut}(C\ell_{p,q})$, $p - q \equiv 0 \pmod 8$, and $p = q = m$. For the matrices of the automorphisms $\widetilde{\ }$ and $\widetilde{\star}$, we have the following two possibilities:

(a) $\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m$, $\mathsf{C} = \mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{2m}$.

$$\mathsf{E}^2 = \begin{cases} +\mathsf{I}, & \text{if } m \equiv 0, 1 \pmod 4,\\ -\mathsf{I}, & \text{if } m \equiv 2, 3 \pmod 4;\end{cases} \quad \mathsf{C}^2 = \begin{cases} +\mathsf{I}, & \text{if } m \equiv 0, 3 \pmod 4,\\ -\mathsf{I}, & \text{if } m \equiv 1, 2 \pmod 4.\end{cases} \tag{29}$$

(b) $\mathsf{E} = \mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{2m}$, $\mathsf{C} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m$.

$$\mathsf{E}^2 = \begin{cases} +\mathsf{I}, & \text{if } m \equiv 0, 3 \pmod 4,\\ -\mathsf{I}, & \text{if } m \equiv 1, 2 \pmod 4;\end{cases} \quad \mathsf{C}^2 = \begin{cases} +\mathsf{I}, & \text{if } m \equiv 0, 1 \pmod 4,\\ -\mathsf{I}, & \text{if } m \equiv 2, 3 \pmod 4.\end{cases} \tag{30}$$

In virtue of (3), for the matrix of the automorphism $\star$ we have always $\mathsf{W}^2 = +\mathsf{I}$.

Now, we are in a position to define automorphism groups for the type $p - q \equiv 0 \pmod 8$. First of all, let us consider Abelian groups. In accordance with (27) and (28), the automorphism group is Abelian if $m \equiv 0 \pmod 2$ (W, E, and C commute with each other). In virtue of (15) and (22), the matrix E should be commuted with the first (symmetric) half and anticommuted with the second (skewsymmetric) half of the basis (22). From (23)–(26) it is easy to see that this condition is satisfied only if $\mathsf{E} = \mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{2m}$ and $m \equiv 0 \pmod 2$. Correspondingly, in accordance with (19), the matrix C should be anticommuted with the symmetric half of the basis (22) and commuted with the skewsymmetric half of the same basis. It is obvious that this condition is satisfied only if $\mathsf{C} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m$. Therefore, when $m = p = q$ in accordance with (30), there exist Abelian groups $\mathsf{Aut}_-(C\ell_{p,q}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ with the signature $(+,+,+)$ if $p, q \equiv 0 \pmod 4$, and $\mathsf{Aut}_-(C\ell_{p,q}) \simeq \mathbb{Z}_4$ with the signature $(+,-,-)$ if $p, q \equiv 2 \pmod 4$. Further, in accordance with (27) and (28), the automorphism group is non-Abelian if $m \equiv 1 \pmod 2$. In this case, from (23)–(26) it follows that the matrix E commutes with the symmetric half and anticommutes with the skewsymmetric half of the basis (22) if and only if $\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m$ is a product of m symmetric matrices, $m \equiv 1 \pmod 8$. In its turn, the matrix C anticommutes with the symmetric half and commutes with the skewsymmetric half of the basis (22) if and only if $\mathsf{C} = \mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{2m}$. Therefore, in accordance with (29), there exist non-Abelian groups $\mathsf{Aut}_+(C\ell_{p,q}) \simeq D_4/\mathbb{Z}_2$ with the signature $(+,-,+)$ if $p, q \equiv 3 \pmod 4$, and $\mathsf{Aut}_+(C\ell_{p,q}) \simeq D_4/\mathbb{Z}_2$ with the signature $(+,+,-)$ if $p, q \equiv 1 \pmod 4$.

In addition to the previously considered case $p = q$, the type $p - q \equiv 0 \pmod 8$ also admits two particular cases in relation with the algebras $C\ell_{p,0}$ and $C\ell_{0,q}$. In these cases, a spinbasis is defined as follows

$$\begin{aligned} \mathcal{E}_i^\top &= \mathcal{E}_i \quad \text{for the algebras } C\ell_{8t,0}, \\ \mathcal{E}_i^\top &= -\mathcal{E}_i \quad \text{for the algebras } C\ell_{0,8t}. \end{aligned} \qquad t = 1, 2, \ldots$$

that is, a spinbasis of the algebra $C\ell_{8t,0}$ consists of only symmetric matrices, and that of $C\ell_{0,8t}$ consists of only skewsymmetric matrices. According to (15), for the algebra $C\ell_{p,0}$ the matrix E should commute with all $\mathcal{E}_i$. It is obvious that we cannot take matrix E of the form $\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_s$, where $1 < s < p$, since at $s \equiv 0 \pmod 2$ E and $\mathcal{E}_i$ anticommute, which contradicts with (15), and at $s \equiv 1 \pmod 2$ E and $\mathcal{E}_i$ that belong to E commute with each other, whereas $\mathcal{E}_i$ that do not belong to E anticommute with E, which again contradicts with (15). The case $s = p$ is also excluded, since p is even. Therefore, only one possibility remains, that is, the matrix E is proportional to the unit matrix, $\mathsf{E} \sim \mathsf{I}$. At this point, from (21) it follows that $\mathsf{C} \sim \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p$, and we see that the conditions (19) are satisfied. Thus, the matrices $\mathsf{E} \sim \mathsf{I}$, $\mathsf{C} = \mathsf{EW}$, and W of the fundamental automorphisms $\mathcal{A} \to \widetilde{\mathcal{A}}$, $\mathcal{A} \to \widetilde{\mathcal{A}^\star}$, and $\mathcal{A} \to \mathcal{A}^\star$ of the algebra $C\ell_{p,0}$ $(p \equiv 0 \pmod 8)$ from an Abelian group $\mathsf{Aut}_-(C\ell_{p,0}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$. Further, for the algebras $C\ell_{0,8t}$, in accordance with (15) the matrix E should anticommute with all $\mathcal{E}_i$. It is easy to see that we cannot take matrix E of the form $\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_k$, where $1 < k < q$, since at $k \equiv 0 \pmod 2$ the matrix E and the matrices $\mathcal{E}_i$ that belong to E anticommute with each other, whereas $\mathcal{E}_i$ that do not belong to E commute with E, which contradicts with (15). Inversely, if $k \equiv 1 \pmod 2$, E and $\mathcal{E}_i$ that belong to E commute, but E and $\mathcal{E}_i$ that do not belong to E anticommute, which also contradicts with (15). It is obvious that in this case $\mathsf{E} \sim \mathsf{I}$ is excluded; therefore, $\mathsf{E} \sim \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_q$. In this case, according to (19) the matrix C is proportional to the unit matrix. Thus, the matrices $\mathsf{E} \sim \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_q$, $\mathsf{C} = \mathsf{EW} \sim \mathsf{I}$, and W of the automorphisms $\mathcal{A} \to \widetilde{\mathcal{A}}$, $\mathcal{A} \to \widetilde{\mathcal{A}^\star}$, and $\mathcal{A} \to \mathcal{A}^\star$ of the algebra $C\ell_{0,q}$ ($q \equiv 0 \pmod 8$) from the group $\mathsf{Aut}_-(C\ell_{0,q}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$.

(2) The type $p - q \equiv 2 \pmod 8$, $\mathbb{K} \simeq \mathbb{R}$.

In virtue of the isomorphism $C\ell_{p,q} \simeq \mathsf{M}_{2^{\frac{p+q}{2}}}(\mathbb{R})$ for the type $p - q \equiv 2 \pmod 8$ in accordance with the signature of the algebra $C\ell_{p,q}$, we have the following basis:

$$\mathcal{E}_i^\top = \begin{cases} \mathcal{E}_i, & \text{if } 1 \le i \le p, \\ -\mathcal{E}_i, & \text{if } p+1 \le i \le p+q. \end{cases} \tag{31}$$

Therefore, in this case the matrix of the antiautomorphism $\widetilde{\ }$ is a product of p symmetric matrices ($\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p$) or is a product of q skewsymmetric matrices ($\mathsf{E} = \mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q}$). Let us find permutation conditions of the matrix E with the basis matrices $\mathcal{E}_i$. Let $\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p$, then

$$\begin{aligned} \mathsf{E}\mathcal{E}_i &= (-1)^{p-i}\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_{i-1}\mathcal{E}_{i+1}\cdots\mathcal{E}_p, \\ \mathcal{E}_i\mathsf{E} &= (-1)^{i-1}\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_{i-1}\mathcal{E}_{i+1}\cdots\mathcal{E}_p, \end{aligned} \quad 1 \le i \le p \tag{32}$$

and

$$\mathsf{E}\mathcal{E}_i = (-1)^p\mathcal{E}_i\mathsf{E}, \quad p+1 \le i \le p+q, \tag{33}$$

that is, at $p \equiv 0 \pmod 2$ the matrix E anticommutes with the symmetric and commutes with the skewsymmetric part the basis (31). Correspondingly, at $p \equiv 1 \pmod 2$ E commutes with the symmetric and anticommutes with the skewsymmetric part of the basis (31).

Analogously, let $\mathsf{E} = \mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q}$, then

$$\mathsf{E}\mathcal{E}_i = (-1)^q\mathcal{E}_i\mathsf{E} \quad 1 \le i \le p \tag{34}$$

and

$$\begin{aligned} \mathsf{E}\mathcal{E}_i &= -(-1)^{q-i}\mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{i-1}\mathcal{E}_{i+1}\cdots\mathcal{E}_{p+q}; \\ \mathcal{E}_i\mathsf{E} &= -(-1)^{i-1}\mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{i-1}\mathcal{E}_{i+1}\cdots\mathcal{E}_{p+q}, \end{aligned} \quad p+1 \le i \le p+q \tag{35}$$

that is, at $q \equiv 0 \pmod 2$ the matrix E commutes with the symmetric and anticommutes with the skewsymmetric part of the basis (31). Correspondingly, at $q \equiv 1 \pmod 2$ E anticommutes with the symmetric and commutes with the skewsymmetric part of (31).

Further, permutation conditions of the matrices $\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p$ and W are defined by the following relations:

$$\begin{aligned}\mathsf{EW} &= \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_{p+q} = (-1)^{\frac{p(p-1)}{2}}\mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q},\\ \mathsf{WE} &= \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_{p+q}\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p = (-1)^{\frac{p(p-1)}{2}+pq}\mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q}.\end{aligned} \tag{36}$$

From a comparison $\frac{p(p-1)}{2} + pq \equiv \frac{p(p-1)}{2} \pmod 2$ it follows that the matrices E and W commute with each other if $pq \equiv 0 \pmod 2$ and anticommute if $pq \equiv 1 \pmod 2$. If we take $\mathsf{E} = \mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q}$, then the relations

$$\begin{aligned}\mathsf{EW} &= \mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q}\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_{p+q} = (-1)^{\frac{q(q+1)}{2}+pq}\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p,\\ \mathsf{WE} &= \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_{p+q}\mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q} = (-1)^{\frac{q(q+1)}{2}}\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p\end{aligned} \tag{37}$$

give analogous permutation conditions for E and W ($pq \equiv 0, 1 \pmod 2$). It is obvious that permutation conditions of C (the matrix of the antiautomorphism $\widetilde{\star}$) with the basis matrices $\mathcal{E}_i$ and with W are analogous to the conditions (32)–(35) and (36)–(37), respectively.

Out of dependence on the choice of the matrices E and C, permutation conditions between them are defined by a relation

$$\mathsf{EC} = (-1)^{pq}\mathsf{CE}, \tag{38}$$

that is, E and C commute if $pq \equiv 0 \pmod 2$ and anticommute if $pq \equiv 1 \pmod 2$.

For the squares of the automorphisms $\widetilde{\ }$ and $\widetilde{\star}$ we have following two possibilities:

(a) $\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p$, $\mathsf{C} = \mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q}$.

$$\mathsf{E}^2 = \begin{cases}+\mathsf{I}, & \text{if } p \equiv 0, 1 \pmod 4;\\ -\mathsf{I}, & \text{if } p \equiv 2, 3 \pmod 4,\end{cases}\quad \mathsf{C}^2 = \begin{cases}+\mathsf{I}, & \text{if } q \equiv 0, 3 \pmod 4;\\ -\mathsf{I}, & \text{if } q \equiv 1, 2 \pmod 4).\end{cases} \tag{39}$$

(b) $\mathsf{E} = \mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q}$, $\mathsf{C} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p$.

$$\mathsf{E}^2 = \begin{cases}+\mathsf{I}, & \text{if } q \equiv 0, 3 \pmod 4;\\ -\mathsf{I}, & \text{if } q \equiv 1, 2 \pmod 4,\end{cases}\quad \mathsf{C}^2 = \begin{cases}+\mathsf{I}, & \text{if } p \equiv 0, 1 \pmod 4;\\ -\mathsf{I}, & \text{if } p \equiv 2, 3 \pmod 4).\end{cases} \tag{40}$$

For the type $p - q \equiv 2 \pmod 8$ in virtue of (3) a square of the matrix W is always equal to $-\mathsf{I}$.

Now, let us consider automorphism groups for the type $p - q \equiv 2 \pmod 8$. In accordance with (36)–(38), the automorphism group $\mathsf{Aut}(C\ell_{p,q})$ is Abelian if $pq \equiv 0 \pmod 2$. Further, in virtue of (15) and (31), the matrix of the antiautomorphism$\widetilde{\ }$should commute with the symmetric part of the basis (31) and anticommute with the skewsymmetric part of the same basis. From (32)–(35), it is easy to see that this condition is satisfied at $pq \equiv 0 \pmod 2$ if and only if $\mathsf{E} = \mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q}$ is a product of q skewsymmetric matrices (recall that for the type $p-q \equiv 2 \pmod 8$, the numbers p and q are both even or both odd). Correspondingly, in accordance with (19), the matrix C should anticommute with the symmetric part of the basis (31) and commute with the skewsymmetric part of the same basis. It is obvious that this requirement is satisfied if and only if $\mathsf{C} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p$ is a product of p symmetric matrices. Thus in accordance with (40), there exist Abelian groups $\mathsf{Aut}_-(C\ell_{p,q}) \simeq \mathbb{Z}_4$ with the signature $(-,-,+)$ if $p \equiv 0 \pmod 4$ and $q \equiv 2 \pmod 4$ and with the signature $(-,+,-)$ if $p \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 4$. Further, according to (36)–(38), the automorphism group is non-Abelian if $pq \equiv 1 \pmod 2$. In this case, from (32)–(35) it follows that the matrix of the antiautomorphism $\widetilde{\ }$ commutes with the symmetric part of the basis (31) and anticommutes with the skewsymmetric part if and only if $\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p$ is a product of p symmetric matrices. In its turn, the matrix C anticommutes with the symmetric part of the basis (31) and commutes with the skewsymmetric part of the same basis if and only if $\mathsf{C} = \mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q}$. Therefore in accordance with (39), there exist non-Abelian groups $\mathsf{Aut}_+(C\ell_{p,q}) \simeq Q_4/\mathbb{Z}_2$ with the signature $(-,-,-)$ if $p \equiv 3 \pmod 4$ and $q \equiv 1 \pmod 4$ $\mathsf{Aut}_+(C\ell_{p,q}) \simeq Q_4/\mathbb{Z}_2$ and with the signature $(-,+,+)$ if $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$.

(3) The type $p-q \equiv 6 \pmod 8$, $\mathbb{K} \simeq \mathbb{H}$.

First of all, over the ring $\mathbb{K} \simeq \mathbb{H}$ there exists no fixed basis of the form (22) or (31) for the matrices $\mathcal{E}_i$. In general, a number of the skewsymmetric matrices does not coincide with a number of matrices with the negative square ($\mathcal{E}_j^2 = -\mathsf{I}$) as it takes place for the types $p-q \equiv 0, 2 \pmod 8$. Thus, the matrix E is a product of skewsymmetric matrices $\mathcal{E}_j$, among which there are matrices with positive and negative squares, or E is a product of symmetric matrices $\mathcal{E}_i$, among which also there are matrices with $(+)$ and $(-)$ squares. Let k be a number of the skewsymmetric matrices $\mathcal{E}_j$ of a spinbasis of the algebra $C\ell_{p,q}$, $0 \leq k \leq p+q$. Among the matrices $\mathcal{E}_j$, l have $(+)$-square and t matrices have $(-)$-square. Let $0 < k \leq p+q$ and let $\mathsf{E} = \mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}$ be a matrix of the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$. Then, permutation conditions of the matrix E with the matrices $\mathcal{E}_{i_r}$ of the symmetric part $(0 < r \leq p+q-k)$ and with the matrices $\mathcal{E}_{j_u}$ of the skewsymmetric part $(0 < u \leq k)$ have the respective form

$$\mathsf{E}\mathcal{E}_{i_r} = (-1)^k\mathcal{E}_{i_r}\mathsf{E} \quad 0 < r \leq p+q-k, \tag{41}$$

$$\left.\begin{aligned}\mathsf{E}\mathcal{E}_{j_u} &= (-1)^{k-u}\sigma(j_u)\mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_{u-1}}\mathcal{E}_{j_{u+1}}\cdots\mathcal{E}_{j_k},\\ \mathcal{E}_{j_u}\mathsf{E} &= (-1)^{u-1}\sigma(j_u)\mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_{u-1}}\mathcal{E}_{j_{u+1}}\cdots\mathcal{E}_{j_k},\end{aligned}\right. \quad 0 < u \leq k \tag{42}$$

that is, at $k \equiv 0 \pmod 2$ the matrix E commutes with the symmetric and anticommutes with the skewsymmetric part of the spinbasis. Correspondingly, at

$k \equiv 1 \pmod 2$, E anticommutes with the symmetric and commutes with the skewsymmetric part. Further, let $\mathsf{E} = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}$ be a product of the symmetric matrices, then

$$\begin{aligned} \mathsf{E}\mathcal{E}_{i_r} &= (-1)^{p+q-k}\sigma(i_r)\mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{r-1}}\mathcal{E}_{i_{r+1}}\cdots\mathcal{E}_{i_{p+q-k}}, \\ \mathcal{E}_{i_r}\mathsf{E} &= \ (-1)^{r-1}\sigma(i_r)\mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{r-1}}\mathcal{E}_{i_{r+1}}\cdots\mathcal{E}_{i_{p+q-k}}, \end{aligned} \quad 0 < r \le p+q-k \qquad (43)$$

$$\mathsf{E}\mathcal{E}_{j_u} = (-1)^{p+q-k}\mathcal{E}_{j_u}\mathsf{E}, \quad 0 < u \le k \qquad (44)$$

that is, at $p+q-k \equiv 0 \pmod 2$ the matrix E anticommutes with the symmetric and commutes with the skewsymmetric part of the spinbasis. Correspondingly, at $p+q-k \equiv 1 \pmod 2$ E commutes with the symmetric and anticommutes with the skewsymmetric part. It is easy to see that permutation conditions of the matrix C with the basis matrices $\mathcal{E}_i$ coincide with (41)–(44).

For the permutation conditions of the matrices $\mathsf{W} = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}\mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}$, $\mathsf{E} = \mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}$, and $\mathsf{C} = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}$ we have

$$\begin{aligned} \mathsf{EW} &= (-1)^{\frac{k(k-1)}{2}+t+k(p+q-k)}\mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}, \\ \mathsf{WE} &= (-1)^{\frac{k(k-1)}{2}+t}\mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}. \end{aligned} \qquad (45)$$

$$\mathsf{EC} = (-1)^{k(p+q-k)}\mathsf{CE}. \qquad (46)$$

Hence it follows that the matrices W, E, and C commute at $k(p+q-k) \equiv 0 \pmod 2$ and anticommute at $k(p+q-k) \equiv 1 \pmod 2$. It is easy to verify that permutation conditions for the matrices $\mathsf{E} = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}$ and $\mathsf{C} = \mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}$ would be the same.

In accordance with (15), (19), (41)–(44), and also with (45)–(46), the Abelian automorphism groups for the type $p-q \equiv 6 \pmod 8$ exist only if $\mathsf{E} = \mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}$ and $\mathsf{C} = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}$, $k \equiv 0 \pmod 2$. Let l and t be the quantities of the matrices in the product $\mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}$, which have $(+)$ and $(-)$-squares, respectively, and also let h and g be the quantities of the matrices with the same meaning in the product $\mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}$. Then, the group $\mathsf{Aut}_-(C\ell_{p,q}) \simeq \mathbb{Z}_4$ with the signature $(-,+,-)$ exists if $l-t \equiv 0, 1, 4, 5 \pmod 8$ and $h-g \equiv 2, 3, 6, 7 \pmod 8$ (recall that for the type $p-q \equiv 6 \pmod 8$ we have $\mathsf{W}^2 = -\mathsf{I}$), and also, the group $\mathsf{Aut}_-(C\ell_{p,q}) \simeq \mathbb{Z}_4$ with the signature $(-,-,+)$ exists if $l-t \equiv 2, 3, 6, 7 \pmod 8$ and $h-g \equiv 0, 1, 4, 5 \pmod 8$. Further, from (15), (19), and (41)–(46), it follows that the non-Abelian automorphism groups exist only if $\mathsf{E} = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}$ and $\mathsf{C} = \mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}$, $k \equiv 1 \pmod 2$. At this point the group $\mathsf{Aut}_+(C\ell_{p,q}) \simeq Q_4/\mathbb{Z}_2$ with the signature $(-,-,-)$ exists if $h-g \equiv 2, 3, 6, 7 \pmod 8$ and $l-t \equiv 2, 3, 6, 7 \pmod 8$. Correspondingly, the group $\mathsf{Aut}_+(C\ell_{p,q}) \simeq D_4/\mathbb{Z}_2$ with the signature $(-,+,+)$ exists if $h-g \equiv 0, 1, 4, 5 \pmod 8$ and $l-t \equiv 0, 1, 4, 5 \pmod 8$. In absence of the skewsymmetric matrices $k = 0$, the spinbasis of $C\ell_{p,q}$ contains only symmetric matrices. In this case, from (15), it follows that the matrix of the antiautomorphism $\mathcal{A} \to \widetilde{\mathcal{A}}$ should commute with all the basis matrices. It is obvious that this condition is satisfied if and only if E is proportional to the unit matrix. At this point, from (21), it follows that $\mathsf{C} \sim \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_{p+q}$ and we see that condition (19) is satisfied. Thus, we have the Abelian group $\mathsf{Aut}_-(C\ell_{p,q}) \simeq \mathbb{Z}_4$ with the signature $(-,+,-)$. In the other degenerate case $k = p+q$, the spinbasis of $C\ell_{p,q}$ contains only skewsymmetric matrices; therefore, the matrix E should anticommute with all the basis matrices. This condition is satisfied if and only if $\mathsf{E} \sim \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_{p+q}$. In its turn, the matrix C commutes with all the basis matrices if and only if $\mathsf{C} \sim \mathsf{I}$. It is easy to see that in this case we have the group $\mathsf{Aut}_-(C\ell_{p,q}) \simeq \mathbb{Z}_4$ with the signature $(-,-,+)$.

(4) The type $p - q \equiv 4 \pmod 8$, $\mathbb{K} \simeq \mathbb{H}$.

It is obvious that a proof for this type is analogous to the case $p - q \equiv 6 \pmod 8$, where also $\mathbb{K} \simeq \mathbb{H}$. For the type $p - q \equiv 4 \pmod 8$ we have $\mathsf{W}^2 = +\mathsf{I}$. As well as for the type $p - q \equiv 6 \pmod 8$, the Abelian groups exist only if $\mathsf{E} = \mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}$ and $\mathsf{C} = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}$, $k \equiv 0 \pmod 2$. At this point the group $\mathsf{Aut}_-(C\ell_{p,q}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ with $(+,+,+)$ exists if $l - t,\ h - g \equiv 0, 1, 4, 5 \pmod 8$, and also the group $\mathsf{Aut}_-(C\ell_{p,q}) \simeq \mathbb{Z}_4$ with $(+,-,-)$ exists if $l - t, h - g \equiv 2, 3, 6, 7 \pmod 8$. Correspondingly, the non-Abelian group exist only if E is a product of k skewsymmetric matrices and C is a product of $p + q - k$ symmetric matrices, $k \equiv 1 \pmod 2$. The group $\mathsf{Aut}_+(C\ell_{p,q}) \simeq D_4/\mathbb{Z}_2$ with $(+,-,+)$ exists if $h - g \equiv 2, 3, 6, 7 \pmod 8$, $l - t \equiv 0, 1, 4, 5 \pmod 8$, and the group $\mathsf{Aut}_+(C\ell_{p,q}) \simeq D_4/\mathbb{Z}_2$ with $(+,+,-)$ exists if $h - g \equiv 0, 1, 4, 5 \pmod 8$, $l - t \equiv 2, 3, 6, 7 \pmod 8$. For the type $p - q \equiv 4 \pmod 8$ both the degenerate cases $k = 0$ and $k = p + q$ give rise to the group $\mathsf{Aut}_-(C\ell_{p,q}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$.

(5) The type $p - q \equiv 1 \pmod 8$, $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$.

In this case a dimensionality $p + q$ is odd and the algebra $C\ell_{p,q}$ is semi-simple. Over the ring $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$ the algebras of this type decompose into a direct sum of two subalgebras with even dimensionality. At this point there exist two types of decomposition (Porteous, 1969; Rashevskii, 1957):

$$C\ell_{p,q} \simeq C\ell_{p,q-1} \oplus C\ell_{p,q-1}, \tag{47}$$

$$C\ell_{p,q} \simeq C\ell_{q,p-1} \oplus C\ell_{q,p-1}, \tag{48}$$

where each algebra $C\ell_{p,q-1}$ ($C\ell_{q,p-1}$) is obtained by means of either of the two central idempotents $\frac{1}{2}(1 \pm \mathbf{e}_1\mathbf{e}_2\cdots\mathbf{e}_{p+q})$ and isomorphisms

$$C\ell^+_{p,q} \simeq C\ell_{p,q-1}, \tag{49}$$

$$C\ell^+_{p,q} \simeq C\ell_{q,p-1}. \tag{50}$$

In general, the structure of the ring $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$ in virtue of the decompositions (47)–(48) and isomorphisms (49)–(50) admits all eight kinds of the automorphism groups, since the subalgebras in the direct sums (47)–(48) have the type $p - q \equiv 2 \pmod 8$ or the type $p - q \equiv 0 \pmod 8$. More precisely, for the algebras $C\ell_{0,q}$ of the type $p - q \equiv 1 \pmod 8$, the subalgebras in the direct sum (47) have the type $p - q \equiv 2 \pmod 8$ and only this type; therefore, in accordance with previously obtained conditions for the type $p - q \equiv 2 \pmod 8$, we have four and only four kinds of the automorphism groups with the signatures $(-, -, +)$, $(-, -, +)$ and $(-, -, -)$, $(-, +, +)$. Further, for the algebra $C\ell_{p,0}$ ($p - q \equiv 1 \pmod 8$), the subalgebras in the direct sum (48) have the type $p - q \equiv 0 \pmod 8$; therefore, in this case there exist four and only four kinds of the automorphism groups with the signatures $(+, +, +)$, $(+, -, -)$ and $(+, -, +)$, $(+, +, -)$. In a general case, $C\ell_{p,q}$, the type $p - q \equiv 1 \pmod 8$ admits all eight kinds of the automorphism groups.

(6) The type $p - q \equiv 5 \pmod 8$, $\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}$.

In this case the algebra $C\ell_{p,q}$ is also semi-simple and, therefore, we have decompositions of the form (47)–(48). By analogy with the type $p - q \equiv 1 \pmod 8$, a structure of the double quaternionic ring $\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}$ in virtue of the decompositions (47)–(48) and isomorphisms (49)–(50), also admits, in a general case, all eight kinds of the automorphism groups, since the subalgebras in the direct sums (47)–(48) have the type $p - q \equiv 6 \pmod 8$ or the type $p - q \equiv 4 \pmod 8$. More precisely, for the algebras $C\ell_{0,q}$ of the type $p - q \equiv 5 \pmod 8$, the subalgebras in the direct sum (47) have the type $p - q \equiv 6 \pmod 8$ and only this type; therefore, in accordance with previously obtained results for the quaternionic rings we have four and only four kinds of the automorphism groups with the signatures $(-, +, -)$, $(-, -, +)$ and $(-, -, -)$, $(-, +, +)$. Analogously, for the algebras $C\ell_{p,0}$ ($p - q \equiv 5 \pmod 8$), the subalgebras in the direct sum (48) have the type $p - q \equiv 4 \pmod 8$; therefore, in this case there exist four and only four kinds of the automorphism groups with the signatures $(+, +, +)$, $(+, -, -)$ and $(+, -, +)$, $(+, +, -)$. In a general case, $C\ell_{p,q}$, the type $p - q \equiv 5 \pmod 8$ admits all eight kinds of the automorphism groups.

(7) The type $p - q \equiv 3 \pmod 8$, $\mathbb{K} \simeq \mathbb{C}$.

For this type a center $\mathbf{Z}$ of the algebra $C\ell_{p,q}$ consists of the unit and the volume element $\omega = \mathbf{e}_1\mathbf{e}_2 \cdots \mathbf{e}_{p+q}$, since $p + q$ is odd and the element ω commutes with all the basis elements of the algebra $C\ell_{p,q}$. Moreover, $\omega^2 = -1$, hence it follows that $\mathbf{Z} \simeq \mathbb{R} \oplus i\mathbb{R}$. Thus, for the algebras $C\ell_{p,q}$ of the type $p - q \equiv 3 \pmod 8$, there exists an isomorphism

$$C\ell_{p,q} \simeq \mathbb{C}_{n-1}, \tag{51}$$

where $n = p + q$. It is easy to see that the algebra $\mathbb{C}_{n-1} = \mathbb{C}_{2m}$ in (51) is a complex algebra with even dimensionality, where m is either even or odd. More precisely, the number m is even if $p \equiv 0 \pmod 2$ and $q \equiv 1 \pmod 2$, and odd if $p \equiv 1 \pmod 2$ and $q \equiv 0 \pmod 2$. In accordance with Theorem 2, at $m \equiv 0 \pmod 2$ the algebra $\mathbb{C}_{2m}$ admits the Abelian group $\mathsf{Aut}_-(\mathbb{C}_{2m}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ with $(+,+,+)$, and at $m \equiv 1 \pmod 2$ the non-Abelian group $\mathsf{Aut}_+(\mathbb{C}_{2m}) \simeq Q_4/\mathbb{Z}_2$ with $(-,-,-)$. Hence it follows the statement of the theorem for this type.

(8) The type $p - q \equiv 7 \pmod 8$, $\mathbb{K} \simeq \mathbb{C}$.

It is obvious that for this type the isomorphism (51) also takes places. Therefore, the type $p - q \equiv 7 \pmod 8$ admits the group $\mathsf{Aut}_-(C\ell_{p,q}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ if $p \equiv 0 \pmod 2$ and $q \equiv 1 \pmod 2$, and also the group $\mathsf{Aut}_+(C\ell_{p,q}) \simeq Q_4/\mathbb{Z}_2$ if $p \equiv 1 \pmod 2$ and $q \equiv 0 \pmod 2$. $\square$

Corollary 1. *The matrices* E *and* C *of the antiautomorphisms* $\mathcal{A} \to \widetilde{\mathcal{A}}$ *and* $\mathcal{A} \to \widetilde{\mathcal{A}^\star}$ *over the field* $\mathbb{F} = \mathbb{R}$ *satisfy the following conditions*

$$\mathsf{E}^\top = (-1)^{\frac{m(m-1)}{2}}\mathsf{E}, \quad \mathsf{C}^\top = (-1)^{\frac{m(m+1)}{2}}\mathsf{C}, \tag{52}$$

that is, E *is symmetric if* $m \equiv 0, 1 \pmod 4$ *and skewsymmetric if* $m \equiv 2, 3 \pmod 4$. *Correspondingly,* C *is symmetric if* $m \equiv 0, 3 \pmod 4$ *and skewsymmetric if* $m \equiv 1, 2 \pmod 4$.

Proof: Let us consider first the types with the ring $\mathbb{K} \simeq \mathbb{R}$. As follows from Theorem 4, the type $p - q \equiv 0 \pmod 8$ admits the Abelian automorphism groups ($\mathsf{EC} = \mathsf{CE}$) if E is the product of q skewsymmetric matrices ($q \equiv 0, 2 \pmod 4$) and C is the product of p symmetric matrices ($p \equiv 0, 2 \pmod 4$). Therefore,

$$\begin{aligned}\mathsf{E}^\top &= (\mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{2m})^\top = \mathcal{E}_{2m}^\top\cdots\mathcal{E}_{m+2}^\top\mathcal{E}_{m+1}^\top \\ &= (-\mathcal{E}_{2m})\cdots(-\mathcal{E}_{m+2})(-\mathcal{E}_{m+1}) \\ &= \mathcal{E}_{2m}\cdots\mathcal{E}_{m+2}\mathcal{E}_{m+1} = (-1)^{\frac{q(q-1)}{2}}\mathsf{E},\end{aligned} \tag{53}$$

$$\mathsf{C}^\top = (\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m)^\top = \mathcal{E}_m^\top\cdots\mathcal{E}_2^\top\mathcal{E}_1^\top = \mathcal{E}_m\cdots\mathcal{E}_2\mathcal{E}_1 = (-1)^{\frac{p(p-1)}{2}}\mathsf{C}. \tag{54}$$

Further, the type $p - q \equiv 0 \pmod 8$ admits the non-Abelian automorphism groups ($\mathsf{EC} = -\mathsf{CE}$) if E is the product of p symmetric matrices ($p \equiv 1, 3 \pmod 4$) and C is the product of q-skewsymmetric matrices ($q \equiv 1, 3 \pmod 4$). In this case, we have

$$\mathsf{E}^\top = (\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m)^\top = \mathcal{E}_m^\top\cdots\mathcal{E}_2^\top\mathcal{E}_1^\top = \mathcal{E}_m\cdots\mathcal{E}_2\mathcal{E}_1 = (-1)^{\frac{p(p-1)}{2}}\mathsf{E}, \tag{55}$$

$$\begin{aligned}\mathsf{C}^\top &= (\mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_2)^\top = \mathcal{E}_{2m}^\top\cdots\mathcal{E}_{m+2}^\top\mathcal{E}_{m+1}^\top \\ &= (-\mathcal{E}_{2m})\cdots(-\mathcal{E}_{m+2})(-\mathcal{E}_{m+1}) \\ &= -\mathcal{E}_{2m}\cdots\mathcal{E}_{m+2}\mathcal{E}_{m+1} = -(-1)^{\frac{q(q-1)}{2}}\mathsf{C}.\end{aligned} \tag{56}$$

In the degenerate case $C\ell_{p,0}$, $p \equiv 0 \pmod 8$, we have $\mathsf{E} \sim \mathsf{I}$ and $\mathsf{C} \sim \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p$. Therefore, E is always symmetric and $\mathsf{C}^\top = (-1)^{\frac{p(p-1)}{2}}\mathsf{C}$. In the other degenerate case $C\ell_{0,q}$, $q \equiv 0 \pmod 8$, we have $\mathsf{E} \sim \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_q$ and $\mathsf{C} \sim \mathsf{I}$; therefore, $\mathsf{E}^\top = (-1)^{\frac{q(q-1)}{2}}\mathsf{E}$ and C is always symmetric.

Since for the type $p - q \equiv 0 \pmod 8$ we have $p = q = m$, or $m = p$ and $m = q$ for the degenerate cases (both degenerate cases correspond to the Abelian group $\mathbb{Z}_2 \otimes \mathbb{Z}_2$), it is easy to see that the formulas (53) and (55) coincide with the first formula of (52). For the matrix C, we can unite the formulas (54) and (56) into the formula which coincides with the second formula of (52). Indeed, the factor $(-1)^{\frac{m(m+1)}{2}}$ does not change sign in $\mathsf{C}^\top = (-1)^{\frac{m(m+1)}{2}}\mathsf{C}$ when m is even, and changes sign when m is odd, which is equivalent to both formulas (54) and (56).

Further, the following real type $p - q \equiv 2 \pmod 8$ admits the Abelian automorphism groups if $\mathsf{E} = \mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q}$ and $\mathsf{C} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p$, where p and $q \equiv 0, 2 \pmod 4$. Therefore,

$$\begin{aligned}\mathsf{E}^\top &= (\mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q})^\top = \mathcal{E}_{p+q}^\top\cdots\mathcal{E}_{p+2}^\top\mathcal{E}_{p+1}^\top\\ &= (-\mathcal{E}_{p+q})\cdots(-\mathcal{E}_{p+2})(-\mathcal{E}_{p+1})\\ &= \mathcal{E}_{p+q}\cdots\mathcal{E}_{p+2}\mathcal{E}_{p+1} = (-1)^{\frac{q(q-1)}{2}}\mathsf{E},\end{aligned} \tag{57}$$

$$\mathsf{C}^\top = (\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p)^\top = \mathcal{E}_p^\top\cdots\mathcal{E}_2^\top\mathcal{E}_1^\top = \mathcal{E}_p\cdots\mathcal{E}_2\mathcal{E}_1 = (-1)^{\frac{p(p-1)}{2}}\mathsf{C}. \tag{58}$$

Correspondingly, the type $p - q \equiv 2 \pmod 8$ admits the non-Abelian automorphism groups if $\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p$ and $\mathsf{C} = \mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q}$, where p and $q \equiv 1, 3 \pmod 4$. In this case, we have

$$\mathsf{E}^\top = (\mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_p)^\top = \mathcal{E}_p^\top\cdots\mathcal{E}_2^\top\mathcal{E}_1^\top = \mathcal{E}_p\cdots\mathcal{E}_2\mathcal{E}_1 = (-1)^{\frac{p(p-1)}{2}}\mathsf{E}, \tag{59}$$

$$\begin{aligned}\mathsf{C}^\top &= (\mathcal{E}_{p+1}\mathcal{E}_{p+2}\cdots\mathcal{E}_{p+q})^\top = \mathcal{E}_{p+q}^\top\cdots\mathcal{E}_{p+2}^\top\mathcal{E}_{p+1}^\top\\ &= (-\mathcal{E}_{p+q})\cdots(-\mathcal{E}_{p+2})(-\mathcal{E}_{p+1})\\ &= -\mathcal{E}_{p+q}\cdots\mathcal{E}_{p+2}\mathcal{E}_{p+1} = -(-1)^{\frac{q(q-1)}{2}}\mathsf{C}.\end{aligned} \tag{60}$$

It is easy to see that formulas (57)–(60) are similar to the formulas (53)–(56) and, therefore, the conditions (52) hold for the type $p - q \equiv 2 \pmod 8$.

Analogously, the quaternionic types $p - q \equiv 4, 6 \pmod 8$ admit the Abelian automorphism groups if $\mathsf{E} = \mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}$ and $\mathsf{C} = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}$, where k and $p + q - k$ are even (Theorem 4). Transposition of these matrices gives

$$\begin{aligned}\mathsf{E}^\top &= (\mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k})^\top = \mathcal{E}_{j_k}^\top\cdots\mathcal{E}_{j_2}^\top\mathcal{E}_{j_1}^\top\\ &= (-\mathcal{E}_{j_k})\cdots(-\mathcal{E}_{j_2})(-\mathcal{E}_{j_1})\\ &= \mathcal{E}_{j_k}\cdots\mathcal{E}_{j_2}\mathcal{E}_{j_1} = (-1)^{\frac{k(k-1)}{2}}\mathsf{E},\end{aligned} \tag{61}$$

$$\begin{aligned}\mathsf{C}^{\top} &= \left(\mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}\right)^{\top} = \mathcal{E}^{\top}_{i_{p+q-k}}\cdots\mathcal{E}^{\top}_{i_2}\mathcal{E}^{\top}_{i_1} = \mathcal{E}_{i_{p+q-k}}\cdots\mathcal{E}_{i_2}\mathcal{E}_{i_1}\\ &= (-1)^{\frac{(p+q-k)(p+q-k-1)}{2}}\mathsf{C}.\end{aligned} \tag{62}$$

The non-Abelian automorphism groups take place for the types $p - q \equiv 4, 6 \pmod 8$ if $\mathsf{E} = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}$ and $\mathsf{C} = \mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}$, where k and $p+q-k$ are odd. In this case we have

$$\begin{aligned}\mathsf{E}^{\top} &= \left(\mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_{p+q-k}}\right)^{\top} = \mathcal{E}^{\top}_{i_{p+q-k}}\cdots\mathcal{E}^{\top}_{i_2}\mathcal{E}^{\top}_{i_1}\\ &= \mathcal{E}_{p+q-k}\cdots\mathcal{E}_{i_2}\mathcal{E}_{i_1} = (-1)^{\frac{(p+q-k)(p+q-k-1)}{2}}\mathsf{E},\end{aligned} \tag{63}$$

$$\begin{aligned}\mathsf{C}^{\top} &= \left(\mathcal{E}_{j_1}\mathcal{E}_{j_2}\cdots\mathcal{E}_{j_k}\right)^{\top} = \mathcal{E}^{\top}_{j_k}\cdots\mathcal{E}^{\top}_{j_2}\mathcal{E}^{\top}_{j_1} = \left(-\mathcal{E}_{j_k}\right)\cdots\left(-\mathcal{E}_{j_2}\right)\left(-\mathcal{E}_{j_1}\right)\\ &= -\mathcal{E}_{j_k}\cdots\mathcal{E}_{j_1}\mathcal{E}_{j_1} = -(-1)^{\frac{k(k-1)}{2}}\mathsf{C}.\end{aligned} \tag{64}$$

As it takes place for these two types considered here, we again come to the same situation. Therefore, the conditions (52) hold for the quaternionic types $p - q \equiv 4, 6 \pmod 8$.

In virtue of the isomorphism (51) and Theorem 4, the matrices E and C for the types $p - q \equiv 3, 7 \pmod 8$ with the ring $\mathbb{K} \simeq \mathbb{C}$ have the following form: $\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m$, $\mathsf{C} = \mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{2m}$ if $m \equiv 1 \pmod 2$ ($\mathsf{EC} = -\mathsf{CE}$) and $\mathsf{E} = \mathcal{E}_{m+1}\mathcal{E}_{m+2}\cdots\mathcal{E}_{2m}$, $\mathsf{C} = \mathcal{E}_1\mathcal{E}_2\cdots\mathcal{E}_m$ if $m \equiv 0 \pmod 2$ ($\mathsf{EC} = \mathsf{CE}$). It is obvious that for these types the conditions (52) hold.

Finally, for the semi-simple types $p - q \equiv 1, 5 \pmod 8$ in virtue of the decompositions (47)–(48) we have the formulas (53)–(56) or (57)–(60) in case of the ring $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$ ($p - q \equiv 1 \pmod 8$) and the formulas (61)–(64) in case of the ring $\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}$ ($p - q \equiv 5 \pmod 8$). $\square$

An algebraic structure of the discrete transformations is defined by the isomorphism $\{\mathrm{Id}, \star, \widetilde{\ }, \widetilde{\star}\} \simeq \{1, P, T, PT\}$ (Varlamov, 1999). Using (9) or (12), we can apply this structure to the double coverings of the orthogonal group $O(p, q)$. Obviously, in case of the types $p - q \equiv 0, 2, 4, 6 \pmod 8$, it is established directly. Further, in virtue of the isomorphism (51) for the types $p - q \equiv 3, 7 \pmod 8$, we have

$$\mathbf{Pin}(p, q) \simeq \mathbf{Pin}(n - 1, \mathbb{C}),$$

where $n = p + q$. Analogously, for the semi-simple types $p - q \equiv 1, 5 \pmod 8$ in virtue of the decompositions (47)–(48) the algebra $C\ell_{p,q}$ is isomorphic to a direct sum of two mutually annihilating simple ideals $\frac{1}{2}(1 \pm \omega)C\ell_{p,q}$: $C\ell_{p,q} \simeq \frac{1}{2}(1 + \omega)C\ell_{p,q} \oplus \frac{1}{2}(1 - \omega)C\ell_{p,q}$, where $\omega = \mathbf{e}_{12\cdots p+q}$. At this point, each ideal is isomorphic to $C\ell_{p,q-1}$ or $C\ell_{q,p-1}$. Therefore, for the Clifford–Lipschitz groups of these types, we have the following isomorphisms:

$$\mathbf{Pin}(p,q) \simeq \mathbf{Pin}(p,q-1) \bigcup \mathbf{e}_{12\cdots p+q}\mathbf{Pin}(p,q-1),$$
$$\mathbf{Pin}(p,q) \simeq \mathbf{Pin}(q,p-1) \bigcup \mathbf{e}_{12\cdots p+q}\mathbf{Pin}(q,p-1).$$

Theorem 5. *Let* $\mathbf{Pin}^{a,b,c}(p,q)$ *be a double covering of the orthogonal group* $O(p,q)$ *of the real space* $\mathbb{R}^{p,q}$ *associated with the algebra* $C\ell_{p,q}$*. The squares of symbols* $a,b,c \in \{-,+\}$ *correspond to the squares of the elements of a finite group* $\mathsf{Aut}(C\ell_{p,q}) = \{\mathsf{I}, \mathsf{W}, \mathsf{E}, \mathsf{C}\} : a = \mathsf{W}^2$*,* $b = \mathsf{E}^2$*,* $c = \mathsf{C}^2$*, where* W*,* E*, and* C *are the matrices of the fundamental automorphisms* $\mathcal{A} \to \mathcal{A}^\star$*,* $\mathcal{A} \to \widetilde{\mathcal{A}}$*, and* $\mathcal{A} \to \widetilde{\mathcal{A}^\star}$ *of the algebra* $C\ell_{p,q}$*, respectively. Then over the field* $\mathbb{F} = \mathbb{R}$ *in dependence on a division ring structure of the algebra* $C\ell_{p,q}$*, there exist eight double coverings of the orthogonal group* $O(p,q)$*:*

(1) *A non-Cliffordian group*

$$\mathbf{Pin}^{+,+,+}(p,q) \simeq \frac{(\mathbf{Spin}_0(p,q) \odot \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2)}{\mathbb{Z}_2},$$

exists if $\mathbb{K} \simeq \mathbb{R}$ *and the numbers p and q form the type* $p - q \equiv 0 \pmod 8$ *and* $p, q \equiv 0 \pmod 4$*, and also if* $p - q \equiv 4 \pmod 8$ *and* $\mathbb{K} \simeq \mathbb{H}$*. The algebras* $C\ell_{p,q}$ *with the rings* $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$*,* $\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}$ *(*$p - q \equiv 1, 5 \pmod 8$*) admit the group* $\mathbf{Pin}^{+,+,+}(p,q)$ *if in the direct sums there are addendums of the type* $p - q \equiv 0 \pmod 8$ *or* $p - q \equiv 4 \pmod 8$*. The types* $p - q \equiv 3, 7 \pmod 8$*,* $\mathbb{K} \simeq \mathbb{C}$ *admit a non-Cliffordian group* $\mathbf{Pin}^{+,+,+}(p+q-1, \mathbb{C})$ *if* $p \equiv 0 \pmod 2$ *and* $q \equiv 1 \pmod 2$*. Further, non-Cliffordian groups*

$$\mathbf{Pin}^{a,b,c}(p,q) \simeq \frac{(\mathbf{Spin}_0(p,q) \odot (\mathbb{Z}_2 \otimes \mathbb{Z}_4)}{\mathbb{Z}_2},$$

with $(a,b,c) = (+,-,-)$ *exist if* $p - q \equiv 0 \pmod 8$*,* $p, q \equiv 2 \pmod 4$ *and* $\mathbb{K} \simeq \mathbb{R}$*, and also if* $p - q \equiv 4 \pmod 8$ *and* $\mathbb{K} \simeq \mathbb{H}$*. Non-Cliffordian groups with the signatures* $(a,b,c) = (-,+,-)$ *and* $(a,b,c) = (-,-,+)$ *exist over the ring* $\mathbb{K} \simeq \mathbb{R}$ *(*$p - q \equiv 2 \pmod 8$*) if* $p \equiv 2 \pmod 4$*,* $q \equiv 0 \pmod 4$ *and* $p \equiv 0 \pmod 4$*,* $q \equiv 2 \pmod 4$*, respectively, and also these groups exist over the ring* $\mathbb{K} \simeq \mathbb{H}$ *if* $p - q \equiv 6 \pmod 8$*. The algebras* $C\ell_{p,q}$ *with the rings* $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$*,* $\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}$ *(*$p - q \equiv 1, 5 \pmod 8$*) admit the group* $\mathbf{Pin}^{+,-,-}(p,q)$ *if in the direct sums there are addendums of the type* $p - q \equiv 0 \pmod 8$ *or* $p - q \equiv 4 \pmod 8$ *, and also admit the groups* $\mathbf{Pin}^{-,+,-}(p,q)$ *and* $\mathbf{Pin}^{-,-,+}(p,q)$ *if in the direct sums there are addendums of the type* $p - q \equiv 2 \pmod 8$ *or* $p - q \equiv 6 \pmod 8$.

(2) *A Cliffordian group*

$$\mathbf{Pin}^{-,-,-}(p,q) \simeq \frac{(\mathbf{Spin}_0(p,q) \odot Q_4)}{\mathbb{Z}_2},$$

exists if $\mathbb{K} \simeq \mathbb{R}(p-q \equiv 2 \pmod 8)$ *and* $p \equiv 3 \pmod 4$, $q \equiv 1 \pmod 4$, *and also if* $p-q \equiv 6 \pmod 8$ *and* $\mathbb{K} \simeq \mathbb{H}$. *The algebras* $C\ell_{p,q}$ *with the rings* $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$, $\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}$ $(p-q \equiv 1, 5 \pmod 8)$ *admit the group* $\mathbf{Pin}^{-,-,-}(p,q)$ *if in the direct sums there are addendums of the type* $p-q \equiv 2 \pmod 8$ *or* $p-q \equiv 6 \pmod 8$. *The types* $p-q \equiv 3, 7 \pmod 8$, $\mathbb{K} \simeq \mathbb{C}$ *admit a Cliffordian group* $\mathbf{Pin}^{-,-,-}(p+q-1, \mathbb{C})$, *if* $p \equiv 1 \pmod 2$ *and* $q \equiv 0 \pmod 2$. *Further, Cliffordian groups*

$$\mathbf{Pin}^{a,b,c}(p,q) \simeq \frac{(\mathbf{Spin}_0(p,q) \odot D_4)}{\mathbb{Z}_2},$$

with $(a,b,c) = (-,+,+)$ *exist if* $\mathbb{K} \simeq \mathbb{R}$, $(p-q \equiv 2 \pmod 8)$ *and* $p \equiv 1 \pmod 4$, $q \equiv 3 \pmod 4$, *and also if* $p-q \equiv 6 \pmod 8$ *and* $\mathbb{K} \simeq \mathbb{H}$. *Cliffordian groups with the signatures* $(a,b,c) = (+,-,+)$ *and* $(a,b,c) = (+,+,-)$ *exist over the ring* $\mathbb{K} \simeq \mathbb{R}$, $(p-q \equiv 0 \pmod 8)$ *if* $p, q \equiv 3 \pmod 4$ *and* $p, q \equiv 1 \pmod 4$, *respectively, and also these groups exist over the ring* $\mathbb{K} \simeq \mathbb{H}$ *if* $p-q \equiv 4 \pmod 8$. *The algebras* $C\ell_{p,q}$ *with the rings* $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$, $\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}(p-q \equiv 1, 5 \pmod 8)$ *admit the group* $\mathbf{Pin}^{-,+,+}(p,q)$ *if in the direct sums there are addendums of the type* $p-q \equiv 2 \pmod 8$ *or* $p-q \equiv 6 \pmod 8$, *and also admit the groups* $\mathbf{Pin}^{+,-,+}(p,q)$ *and* $\mathbf{Pin}^{+,+,-}(p,q)$ *if in the direct sums there are addendums of the type* $p-q \equiv 0 \pmod 8$ *or* $p-q \equiv 4 \pmod 8$.

4. THE STRUCTURE OF Pin$(p, q) \not\simeq$ Pin(q, p)

It is easy to see that the definitions (9) and (12) are equivalent. Moreover, Salingaros (1981, 1982, 1984) showed that there are isomorphisms $\mathbb{Z}_2 \otimes \mathbb{Z}_2 \simeq C\ell_{1,0}$ and $\mathbb{Z}_4 \simeq C\ell_{0,1}$. Further, since $C\ell^+_{p,q} \simeq C\ell^+_{q,p}$, in accordance with the definition (10), it follows that $\mathbf{Spin}(p,q) \simeq \mathbf{Spin}(q,p)$. On the other hand, since in a general case $C\ell_{p,q} \not\simeq C\ell_{q,p}$, from the definition (9) it follows that $\mathbf{Pin}(p,q) \not\simeq \mathbf{Pin}(q,p)$ (or $\mathbf{Pin}^{a,b,c}(p,q) \not\simeq \mathbf{Pin}^{a,b,c}(q,p)$). In connection with this, some authors (Cahen *et al.*, 1995, 1998; Choquet-Bruhat *et al.*, 1982; De Witt-Morette, 1982, 1989; Friedrich, 1999; Kirby, 1989) used notations $\mathbf{Pin}^+ \simeq \mathbf{Pin}(p,q)$ and $\mathbf{Pin}^- \simeq \mathbf{Pin}(q,p)$. In Theorems 4 and 5, we have established a relation between the signatures

$$(p,q) = (\underbrace{+,+,\dots,+}_{p \text{ times}}, \underbrace{-,-,\dots,-}_{q \text{ times}})$$

of the spaces $\mathbb{R}^{p,q}$, and the signatures (a, b, c) of the automorphism groups of $C\ell_{p,q}$ and corresponding Dąbrowski groups. This relation allows to completely define the structure of the inequality $\mathbf{Pin}(p, q) \not\simeq \mathbf{Pin}(q, p)$ ($\mathbf{Pin}^+ \not\simeq \mathbf{Pin}^-$). Indeed, from (12) and (11), it follows that $\mathbf{Spin}_0(p, q) \simeq \mathbf{Spin}_0(q, p)$, therefore, a nature of the inequality $\mathbf{Pin}(p, q) \not\simeq \mathbf{Pin}(q, p)$ wholly lies in the double covering $C^{a,b,c}$ of the discrete subgroup. For example, in accordance with Theorem 5 for the type $p - q \equiv 2 \pmod 8$ with the division ring $\mathbb{K} \simeq \mathbb{R}$ there exist the groups $\mathbf{Pin}^{a,b,c}(p, q) \simeq \mathbf{Pin}^+$, where double coverings of the discrete subgroup have the form: (1) $C^{-,-,-} \simeq Q_4$, if $p \equiv 3 \pmod 4$ and $q \equiv 1 \pmod 4$; (2) $C^{-,+,+} \simeq D_4$, if $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$; (3) $C^{-,-,+} \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_4$, if $p \equiv 0 \pmod 4$ and $q \equiv 2 \pmod 4$; (4) $C^{-,+,-} \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_4$, if $p \equiv 2 \pmod 4$ and $q \equiv 0 \pmod 4$. Whereas the groups with opposite signature, $\mathbf{Pin}^- \simeq \mathbf{Pin}^{a,b,c}(q, p)$, have the type $q - p \equiv 6 \pmod 8$ with the ring $\mathbb{K} \simeq \mathbb{H}$. In virtue of the more wide ring $\mathbb{K} \simeq \mathbb{H}$, there exists a far greater choice of the discrete subgroups for each concrete kind of $\mathbf{Pin}^{a,b,c}(q, p)$. Thus,

$$\begin{array}{c} \mathbf{Pin}^{a,b,c}(p, q) \not\simeq \mathbf{Pin}^{a,b,c}(q, p) \\ p - q \equiv 2 \pmod 8 \quad q - p \equiv 6 \pmod 8. \end{array}$$

Further, the type $p - q \equiv 1 \pmod 8$ with the ring $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$ in virtue of Theorems 4 and 5 admits the group $\mathbf{Pin}^{a,b,c}(p, q) \simeq \mathbf{Pin}^+$, where the double covering $C^{a,b,c}$ adopts all the eight possible values. Whereas the opposite type $q - p \equiv 7 \pmod 8$ with the ring $\mathbb{K} \simeq \mathbb{C}$ admits the group $\mathbf{Pin}^{a,b,c}(q, p) \simeq \mathbf{Pin}^-$, where for the double covering $C^{a,b,c}$ of the discrete subgroup there are only two possibilities: (1) $C^{+,+,+} \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \mathbb{Z}_2$, if $p \equiv 0 \pmod 2$ and $q \equiv 1 \pmod 2$; (2) $C^{-,-,-} \simeq Q_4$, if $p \equiv 1 \pmod 2$ and $q \equiv 0 \pmod 2$. The analogous situation takes place for the two mutually opposite types $p - q \equiv 3 \pmod 8$ with $\mathbb{K} \simeq \mathbb{C}$ and $q - p \equiv 5 \pmod 8$ with $\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}$. Therefore,

$$\begin{array}{ccc} \mathbf{Pin}^{a,b,c}(p, q) & \not\simeq & \mathbf{Pin}^{a,b,c}(q, p) \\ p - q \equiv 1 \pmod 8 & & q - p \equiv 7 \pmod 8; \\ \mathbf{Pin}^{a,b,c}(p, q) & \not\simeq & \mathbf{Pin}^{a,b,c}(q, p) \\ p - q \equiv 3 \pmod 8 & & q - p \equiv 5 \pmod 8. \end{array}$$

It is easy to see that an opposite type to the type $p - q \equiv 0 \pmod 8$ with the ring $\mathbb{K} \simeq \mathbb{R}$ is the same type $q - p \equiv 0 \pmod 8$. Therefore, in virtue of Theorems 4 and 5 double coverings $C^{a,b,c}$ for the groups $\mathbf{Pin}^{a,b,c}(p, q) \simeq \mathbf{Pin}^+$ and $\mathbf{Pin}^{a,b,c}(q, p) \simeq \mathbf{Pin}^-$ coincide. The same is the situation for the type $p - q \equiv 4 \pmod 8$ with $\mathbb{K} \simeq \mathbb{H}$, which has the opposite type $q - p \equiv 4 \pmod 8$. Thus,

$$\begin{array}{ccc} \mathbf{Pin}^{a,b,c}(p, q) & \simeq & \mathbf{Pin}^{a,b,c}(q, p) \\ p - q \equiv 0 \pmod 8 & & q - p \equiv 0 \pmod 8; \end{array}$$

$$\mathbf{Pin}^{a,b,c}(p,q) \simeq \mathbf{Pin}^{a,b,c}(q,p)$$

$$p - q \equiv 4 \pmod 8 \qquad q - p \equiv 4 \pmod 8.$$

We will call the types $p - q \equiv 0 \pmod 8$ and $p - q \equiv 4 \pmod 8$, which coincide with their opposite types, *neutral types*.

Example. Let us consider a structure of the inequality $\mathbf{Pin}(3,1) \not\simeq \mathbf{Pin}(1,3)$. The groups $\mathbf{Pin}(3,1)$ and $\mathbf{Pin}(1,3)$ are two different double coverings of the general Lorentz group. These groups play an important role in physics (Carlip and De Witt-Morette; 1998, De Witt-Morette and De Witt, 1990; De Witt-Morette and Gwo, 1990; De Witt-Morette *et al.*, 1997). As follows from (9) the group $\mathbf{Pin}(3,1)$ is completely defined in the framework of the Majorana algebra $C\ell_{3,1}$, which has the type $p - q \equiv 2 \pmod 8$ and the division ring $\mathbb{K} \simeq \mathbb{R}$. As noted previously, the structure of the inequality $\mathbf{Pin}(p,q) \not\simeq \mathbf{Pin}(q,p)$ is defined by the double covering $C^{a,b,c}$. From Theorems 4 and 5, it follows that the algebra $C\ell_{3,1} \simeq \mathrm{M}_4(\mathbb{R})$ admits one and only one group $\mathbf{Pin}^{-,-,-}(3,1)$, where a double covering of the discrete subgroup has a form $C^{-,-,-} \simeq Q_4$. Indeed, let us consider a matrix representation of the units of $C\ell_{3,1}$, using the Maple V and the CLIFFORD package developed by Abłamowicz (1996, 1998, 2000). Let $f = \frac{1}{4}(1 + e_1)(1 + e_{34})$ be a primitive idempotent of the algebra $C\ell_{3,1}$ (prestored idempotent for $C\ell_{3,1}$ in CLIFFORD); then a following CLIFFORD command sequence gives:

```
> restart:with(Cliff4):with(double):
> dim := 4:
> eval(makealiases(dim)):
> B := linalg(diag(1,1,1,−1)):
> Clibasis := cbasis(dim):
> data := clidata(B):
> f := data[4]:
> left_sbasis := minimalideal(clibasis,f,'left'):
> Kbasis := Kfield(left_sbasis,f):
> SBgens := left_sbasis[2]:FBgens := Kbasis[2]:
> K_basis := spinorKbasis(SBgens,f,FBgens,'left'):
> for i from 1 to 4 do
```

$$\mathtt{E[i]} := \mathtt{spinorKrepr(e.i.,\ K\_basis[1],\ FBgens,'left')od;}$$

$$E_1 := \begin{bmatrix} Id & 0 & 0 & 0 \\ 0 & -Id & 0 & 0 \\ 0 & 0 & -Id & 0 \\ 0 & 0 & 0 & -Id \end{bmatrix}, \qquad E_2 := \begin{bmatrix} 0 & Id & 0 & 0 \\ Id & 0 & 0 & 0 \\ 0 & 0 & 0 & Id \\ 0 & 0 & Id & 0 \end{bmatrix},$$

$$E_3 := \begin{bmatrix} 0 & 0 & Id & 0 \\ 0 & 0 & 0 & -Id \\ Id & 0 & 0 & 0 \\ 0 & -Id & 0 & 0 \end{bmatrix}, \qquad E_4 := \begin{bmatrix} 0 & 0 & -Id & 0 \\ 0 & 0 & 0 & Id \\ Id & 0 & 0 & 0 \\ 0 & -Id & 0 & 0 \end{bmatrix}. \tag{65}$$

It is easy to see that the matrices (65) build up a basis of the form (31). Since the condition $pq \equiv 1 \pmod 2$ is satisfied for the algebra $C\ell_{3,1}$, the automorphism group $\mathsf{Aut}(C\ell_{3,1})$ is non-Abelian. In accordance with (15), the matrix E should commute with a symmetric part of the basis (65) and anticommute with a skewsymmetric part of (65). In this case, as follows from (32)–(35) and (65), the matrix E is a product of $p = 3$ symmetric matrices, that is,

$$\mathsf{E} = \mathcal{E}_1\mathcal{E}_2\mathcal{E}_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Further, matrices of the automorphisms $\star$ and $\widetilde{\star}$ for the basis (65) have a form

$$\mathsf{W} = \mathcal{E}_1\mathcal{E}_2\mathcal{E}_3\mathcal{E}_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \qquad \mathsf{C} = \mathsf{EW} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Thus, a group of the fundamental automorphisms of the algebra $C\ell_{3,1}$ in the matrix representation is defined by a finite group $\{\mathsf{I}, \mathsf{W}, \mathsf{E}, \mathsf{C}\} \sim \{I, \mathcal{E}_{1234}, \mathcal{E}_{123}, \mathcal{E}_4\}$. The multiplication table of this group has a form

	I	$\mathcal{E}_{1234}$	$\mathcal{E}_{123}$	$\mathcal{E}_4$
I	I	$\mathcal{E}_{1234}$	$\mathcal{E}_{123}$	$\mathcal{E}_4$
$\mathcal{E}_{1234}$	$\mathcal{E}_{1234}$	$-I$	$\mathcal{E}_4$	$-\mathcal{E}_{123}$
$\mathcal{E}_{123}$	$\mathcal{E}_{123}$	$-\mathcal{E}_4$	$-I$	$\mathcal{E}_{1234}$
$\mathcal{E}_4$	$\mathcal{E}_4$	$\mathcal{E}_{123}$	$-\mathcal{E}_{1234}$	$-I$

$\sim$

	I	W	E	C
I	I	W	E	C
W	W	−I	C	−E
E	E	−C	−I	W
C	C	E	−W	−I

. (66)

From the table, it follows that $\mathsf{Aut}_+(C\ell_{3,1}) \simeq \{\mathsf{I}, \mathsf{W}, \mathsf{E}, \mathsf{C}\} \simeq Q_4/\mathbb{Z}_2$ and, therefore, the algebra $C\ell_{3,1}$ admits a Cliffordian group $\mathbf{Pin}^{-,-,-}(3,1)$ (Theorem 5). It is easy to verify that the double covering $C^{-,-,-} \simeq Q_4$ is an invariant fact for the algebra $C\ell_{3,1}$, that is, $C^{-,-,-}$ does not depend on the choice of the matrix representation. Indeed, for each of the two commuting elements of the algebra $C\ell_{3,1}$, there exist four different primitive idempotents that generate four different matrix representations of $C\ell_{3,1}$. The invariability of the previously mentioned fact is easily verified with the help of a procedure `commutingelements` of the CLIFFORD package, which allows to consider in sequence all the possible primitive idempotents of the algebra $C\ell_{3,1}$ and their corresponding matrix representations.

Now, let us consider discrete subgroups of the double covering $\mathbf{Pin}(1,3)$. The group $\mathbf{Pin}(1,3)$, in turn, is completely constructed within the spacetime algebra $C\ell_{3,1}$ that has the opposite (in relation to the Majorana algebra $C\ell_{3,1}$) type $p-q\equiv 6 \pmod 8$ with the division ring $\mathbb{K}\simeq\mathbb{H}$. According to Wedderburn–Artin theorem, in this case there is an isomorphism $C\ell_{3,1}\simeq \mathsf{M}_2(\mathbb{H})$. The following CLIFFORD command sequence allows to find matrix representations of the units of the algebra $C\ell_{3,1}$ for a prestored primitive idempotent $f=\frac{1}{2}(1+e_{14})$:

```
>  restart:with(Cliff4):with(double):                          (67)
>  dim := 4: eval(makealiases(dim):                            (68)
>  B := linalg(diag(1,-1,-1,-1)):                              (69)
>  clibasis := cbasis(dim):                                    (70)
>  data := clidata(B): f := data[4]:                           (71)
>  left_sbasis := minimalideal(clibasis, f,' left'):           (72)
>  Kbasis := Kfield(left_sbasis, f):                           (73)
>  SBgens := left_sbasis [2]: FBgens := Kbasis[2]:             (74)
>  K_basis := spinorKbasis (SBgens, f, FBgens,' left'):        (75)
>  for i from 1 to 4 do                                        (76)

E[i] := spinorKrepr(e.i.,K_basis[1],FBgens,' left')od;         (77)
```

$$
\begin{aligned}
E_1 : &= \begin{bmatrix} 0 & Id \\ Id & 0 \end{bmatrix}, \quad E_2 := \begin{bmatrix} e2 & 0 \\ 0 & -e2 \end{bmatrix}, \\
E_3 : &= \begin{bmatrix} e3 & 0 \\ 0 & -e3 \end{bmatrix}, \quad E_4 := \begin{bmatrix} 0 & -Id \\ Id & 0 \end{bmatrix}.
\end{aligned} \tag{78}
$$

At this point, the division ring $\mathbb{K}\simeq\mathbb{H}$ is generated by a set $\{1,\mathbf{e}_2,\mathbf{e}_3,\mathbf{e}_{23}\}\simeq\{1,\mathbf{i},\mathbf{j},\mathbf{k}\}$, where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are well-known quaternion units. The basis (78) contains three symmetric matrices and one skewsymmetric matrix. Therefore, in accordance with (15) and (41)–(46), the matrix of the antiautomorphism $\mathcal{A}\to\widetilde{\mathcal{A}}$ is a product of symmetric matrices of the basis (78). Thus,

$$
\begin{gathered}
\mathsf{W}=\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3\mathcal{E}_4=\begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}, \quad \mathsf{E}=\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3=\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, \\
\mathsf{C}=\mathsf{EW}=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\end{gathered} \tag{79}
$$

It is easy to verify that a set of the matrices (79) added by the unit matrix forms the non-Abelian group $\mathsf{Aut}_{+}(C\ell_{1,3}) \simeq Q_4/\mathbb{Z}_2$, with a multiplication table of the form (66). Therefore, the spacetime algebra $C\ell_{1,3}$ admits the Cliffordian group $\mathbf{Pin}^{-,-,-}(1,3)$, where a double covering of the discrete subgroup has the form $C^{-,-,-} \simeq Q_4$. However, as follows from Theorem 4, in virtue of the more wide ring $\mathbb{K} \simeq \mathbb{H}$ the group $\mathbf{Pin}^{-,-,-}(1,3)$ is not the only possible group for the algebra $C\ell_{1,3} \simeq \mathsf{M}_2(\mathbb{H})$. Indeed, looking over all the possible commuting elements of the algebra $C\ell_{1,3}$ we find with the help of the procedure `commutingelements` that

```
> L1 := commutingelements(clibasis);
```

$$L1 := [e1] \tag{80}$$

```
> L2 := commutingelements(remove(member,clibasis,L1));
```

$$L2 := [e12] \tag{81}$$

```
> L3 := commutingelements(remove(member,clibasis,[op(L1),
  op(L2)]));
```

$$L3 := [e13] \tag{82}$$

```
> L4 := commutingelements(remove(member,clibasis,[op(L1),
  op(L2),op(L3)]));
```

$$L4 := [e14] \tag{83}$$

```
> L5 := commutingelements(remove(member,clibasis,[op(L1),
  op(L2),op(L3),op(L4)]));
```

$$L5 := [e234] \tag{84}$$

```
> f := cmulQ((1/2)*(Id + e2we3we4);
```

$$f := \frac{1}{2}Id + \frac{1}{2}e234 \tag{85}$$

```
> type(f,primitiveidemp);
```

$$true \tag{86}$$

It is easy to verify that primitive idempotents $\frac{1}{2}(1 \pm \mathbf{e}_1)$, $\frac{1}{2}(1 \pm \mathbf{e}_{12})$, $\frac{1}{2}(1 \pm \mathbf{e}_{13})$, and $\frac{1}{2}(1 \pm \mathbf{e}_{14})$ constructed by means of the commuting elements $\mathbf{e}_1$, $\mathbf{e}_{12}$, $\mathbf{e}_{13}$, and $\mathbf{e}_{14}$ generate matrix representations that give rise to the group $\mathsf{Aut}_{+}(C\ell_{1,3}) \simeq Q_4/\mathbb{Z}_2$. However, the situation changes for the element $\mathbf{e}_{234}$ and the corresponding primitive idempotent $\frac{1}{2}(1 + \mathbf{e}_{234})$ ($\frac{1}{2}(1 - \mathbf{e}_{234})$). Indeed, executing the commands (85) and (86) and subsequently the commands (72)–(77), we find that

```
> for i from 1 to 4 do

      E[i] := spinorKrepr(e.i., K_basis[1], FBgens,' left')od;
```

$$\begin{aligned} E_1 : &= \begin{bmatrix} 0 & Id \\ Id & 0 \end{bmatrix}, \quad E_2 := \begin{bmatrix} e2 & 0 \\ 0 & -e2 \end{bmatrix}, \\ E_3 : &= \begin{bmatrix} e34 & 0 \\ 0 & -e34 \end{bmatrix}, \quad E_4 := \begin{bmatrix} e4 & 0 \\ 0 & -e4 \end{bmatrix}, \end{aligned} \tag{87}$$

where the division ring $\mathbb{K} \simeq \mathbb{H}$ is generated by a set $\{1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_{24}\} \simeq \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. The basis (87) consists of symmetric matrices only. Therefore in accordance with (15), the matrix E should commute with all the matrices of basis (87). It is obvious that this condition is satisfied only if E is proportional to the unit matrix (recall that any element of the automorphism group may be multiplied by an arbitrary factor $\eta \in \mathbb{F}$, in this case $\mathbb{F} = \mathbb{R}$). Further, a set of the matrices $\mathsf{W} = \mathcal{E}_1\mathcal{E}_2\mathcal{E}_3\mathcal{E}_4$, $\mathsf{E} \sim \mathsf{I}$, and $\mathsf{C} = \mathsf{EW}$ added by the unit matrix forms a finite group with the following multiplication table

	I	W	E	C
I	I	W	E	C
W	W	−I	C	−E
E	E	C	I	W
C	C	−E	W	−I

.

As follows from the table, we have in this case the Abelian group $\mathsf{Aut}_-(C\ell_{1,3}) \simeq \mathbb{Z}_4$ with the signature $(-, +, -)$. Thus, the spacetime algebra $C\ell_{1,3}$ admits the group $\mathbf{Pin}^{-,+,-}(1, 3)$, where a double covering of the discrete subgroup has the form $C^{-,+,-} \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_4$.

This fulfilled analysis explicitly shows a difference between the two double coverings $\mathbf{Pin}(3, 1)$ and $\mathbf{Pin}(1, 3)$ of the Lorentz group. Since double coverings of the connected components of both groups $\mathbf{Pin}(3, 1)$ and $\mathbf{Pin}(1, 3)$ are isomorphic, $\mathbf{Spin}_0(3, 1) \simeq \mathbf{Spin}_0(1, 3)$, the nature of difference between them consists in the concrete form and number of the double covering $C^{a,b,c}$ of the discrete subgroups. So, for the Majorana algebra $C\ell_{3,1}$, all the existing primitive idempotents $\frac{1}{4}(1 \pm \mathbf{e}_1)(1 \pm \mathbf{e}_{34})$, $\frac{1}{4}(1 \pm \mathbf{e}_1)(1 \pm \mathbf{e}_{24})$, $\frac{1}{4}(1 \pm \mathbf{e}_2)(1 \pm \mathbf{e}_{14})$, $\frac{1}{4}(1 \pm \mathbf{e}_3)(1 \pm \mathbf{e}_{134})$, and $\frac{1}{4}(1 \pm \mathbf{e}_{34})(1 \pm \mathbf{e}_{234})$ generate 20 matrix representations, each of which gives rise to the double covering $C^{-,-,-} \simeq Q_4$. On the other hand, for the spacetime algebra $C\ell_{1,3}$ primitive idempotents $\frac{1}{2}(1 \pm \mathbf{e}_{14})$, $\frac{1}{2}(1 \pm \mathbf{e}_1)$, $\frac{1}{2}(1 \pm \mathbf{e}_{12})$, and $\frac{1}{2}(1 \pm \mathbf{e}_{13})$ generate eight matrix representations with $C^{-,-,-} \simeq Q_4$, whereas remaining two primitive idempotents $\frac{1}{2}(1 \pm \mathbf{e}_{234})$ generate matrix representations with $C^{-,+,-} \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_4$.

Remark. Physicists commonly use a transition from some given signature to its opposite (signature change) by means of a replacement $\mathcal{E}_i \to i\mathcal{E}_i$ (so-called Wick rotation). However, such a transition is unsatisfactory from a mathematical viewpoint. For example, we can use the replacement $\mathcal{E}_i \to i\mathcal{E}_i$ for a transition from the spacetime algebra $C\ell_{1,3} \simeq \mathsf{M}_2(\mathbb{H})$ to the Majorana algebra $C\ell_{3,1} \simeq \mathsf{M}_4(\mathbb{R})$ since $i \in \mathsf{M}_2(\mathbb{H})$, whereas an inverse transition $C\ell_{3,1} \to C\ell_{1,3}$ can not be performed by the replacement $\mathcal{E}_i \to i\mathcal{E}_i$, since $i \notin \mathsf{M}_4(\mathbb{R})$. The mathematically correct alternative to the Wick rotation is a tilt-transformation introduced by Lounesto (1993). The tilt-transformation is expressed by a map $ab \to a_+b_+ + b_+a_- + b_-a_+ - b_-a_-$, where $a_\pm, b_\pm \in C\ell^\pm_{p,q}$. The further developing of the tilt-transformation and its application for a formulation of physical theories in the spaces with different signatures has been considered in the recent paper by Miralles (in press).

5. DISCRETE TRANSFORMATIONS AND BRAUER–WALL GROUPS

The algebra $C\ell$ is naturally $\mathbb{Z}_2$-graded. Let $C\ell^+$ (correspondingly $C\ell^-$) be a set consisting of all even (correspondingly odd) elements of the algebra $C\ell$. The set $C\ell^+$ is a subalgebra of $C\ell$. It is obvious that $C\ell = C\ell^+ \oplus C\ell^-$, and also $C\ell^+C\ell^+ \subset C\ell^+$, $C\ell^+C\ell^- \subset C\ell^-$, $C\ell^-C\ell^+ \subset C\ell^-$, $C\ell^-C\ell^- \subset C\ell^+$. A degree $\deg \det a$ of the even (correspondingly odd) element $a \in C\ell$ is equal to 0 (correspondingly 1). Let $\mathfrak{A}$ and $\mathfrak{B}$ be the two associative $\mathbb{Z}_2$-graded algebras over the field $\mathbb{F}$; then a multiplication of homogeneous elements $\mathfrak{a}' \in \mathfrak{A}$ and $\mathfrak{b} \in \mathfrak{B}$ in a graded tensor product $\mathfrak{A}\hat{\otimes}\mathfrak{b}$ is defined as follows: $(\mathfrak{A} \otimes \mathfrak{b})(\mathfrak{A}' \otimes \mathfrak{b}') = (-1)^{\deg \mathfrak{b} \deg \mathfrak{A}'}\mathfrak{A}\mathfrak{A}' \otimes \mathfrak{b}\mathfrak{b}'$. The graded tensor product of the two graded central simple algebras is also graded central simple [Wall, 1964, Theorem 2]. The Clifford algebra $C\ell_{p,q}$ is central simple if $p - q \not\equiv 1, 5 \pmod 8$. It is known that for a Clifford algebra with odd dimensionality, the isomorphisms are as follows: $C\ell^+_{p,q+1} \simeq C\ell_{p,q}$ and $C\ell^+_{p+1,q} \simeq C\ell_{q,p}$ (Porteous, 1969; Rashevskii, 1957). Thus, $C\ell^+_{p,q+1}$ and $C\ell^+_{p+1,q}$ are central simple algebras. Further, in accordance with Chevalley Theorem (Chevalley, 1955), for the graded tensor product there is an isomorphism $C\ell_{p,q}\hat{\otimes}C\ell_{p',q'} \simeq C\ell_{p+p',q+q'}$. Two algebras $C\ell_{p,q}$ and $C\ell_{p',q'}$ are said to be of the same class if $p + q' \equiv p' + q \pmod 8$. The graded central simple Clifford algebras over the field $\mathbb{F} = \mathbb{R}$ form eight similarity classes, which, as it is easy to see, coincide with the eight types of algebras $C\ell_{p,q}$. The set of these 8 types (classes) forms a Brauer–Wall group $BW_\mathbb{R}$ (Wall, 1964) that is isomorphic to a cyclic group $\mathbb{Z}_8$. Thus, the algebra $C\ell_{p,q}$ is an element of the Brauer–Wall group, and a group operation is the graded tensor product $\hat{\otimes}$. A cyclic structure of the group $BW_\mathbb{R} \simeq \mathbb{Z}_8$ may be represented on the Trautman diagram (spinorial clock) (Budinich and Trautman, 1987, 1988) (Fig. 1) by means of a transition $C\ell^+_{p,q} \xrightarrow{h} C\ell_{p,q}$ (the round on the diagram is realized by an hour-hand). At this point, the type of the algebra is defined on the diagram by an equality $q - p = h + 8r$, where $h \in \{0, \ldots, 7\}$, $r \in \mathbb{Z}$.

It is obvious that a group structure over $C\ell_{p,q}$, defined by $BW_\mathbb{R} \simeq \mathbb{Z}_8$, immediately relates with the Atiyah–Bott–Shapiro periodicity (Atiyah *et al.*, 1964).

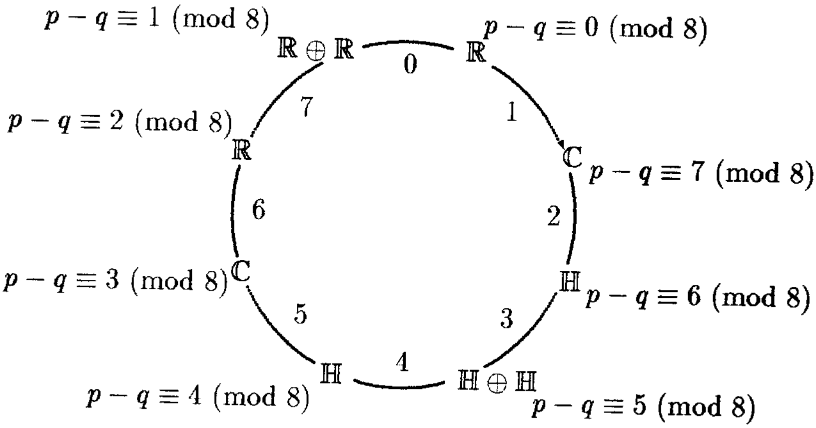

Fig. 1. The Trautman diagram for the Brauer–Wall group $BW_{\mathbb{R}} \simeq \mathbb{Z}_8$.

In accordance with Atiyah (1964), the Clifford algebra over the field $\mathbb{F} = \mathbb{R}$ is modulo 8 periodic: $C\ell_{p+8,q} \simeq C\ell_{p,q} \otimes C\ell_{8,0}(C\ell_{p,q+8} \simeq C\ell_{p,q} \otimes C\ell_{0,8})$.

Coming back to Theorem 4, we see that for each type of algebra $C\ell_{p,q}$ there exists some set of the automorphism groups. If we take into account this relation, then the cyclic structure of a generalized group $BW_{\mathbb{R}}^{a,b,c}$ would look as follows (Fig. 2). First of all, the semi-simple algebras $C\ell_{p,q}$ with the rings $\mathbb{K} \simeq \mathbb{R} \oplus \mathbb{R}$ and $\mathbb{K} \simeq \mathbb{H} \oplus \mathbb{H}$ ($p - q \equiv 1, 5 \pmod 8$) form an axis of the eighth order, which defines the cyclic group $\mathbb{Z}_8$. Further, the neutral types $p - q \equiv 0 \pmod 8$ ($\mathbb{K} \simeq \mathbb{R}$) and $p - q \equiv 4 \pmod 8$ ($\mathbb{K} \simeq \mathbb{H}$), which in common admit the automorphism groups with the signatures $(+, b, c)$, form an axis of the fourth order corresponding to the cyclic group $\mathbb{Z}_4$. Analogously, the two mutually opposite types $p - q \equiv 2 \pmod 8$ ($\mathbb{K} \simeq \mathbb{R}$) and $p - q \equiv 6 \pmod 8$ ($\mathbb{K} \simeq \mathbb{H}$), which in common admit the automorphism groups with the signatures $(-, b, c)$, also form an axis of the fourth order. Finally, the types $p - q \equiv 3, 7 \pmod 8$ ($\mathbb{K} \simeq \mathbb{C}$) with the $(+, +, +)$ and $(-, -, -)$ automorphism groups form an axis of the second order. Therefore, $BW_{\mathbb{R}}^{a,b,c} \simeq \mathbb{Z}_2 \otimes (\mathbb{Z}_4)^2 \otimes \mathbb{Z}_8$, where $(\mathbb{Z}_4)^2 = \mathbb{Z}_4 \otimes \mathbb{Z}_4$.

Further, over the field $\mathbb{F} = \mathbb{C}$, there exist two types of the complex Clifford algebras: $\mathbb{C}_n$ and $\mathbb{C}_{n+1} \simeq \mathbb{C}_n \oplus \mathbb{C}_n$. Therefore, a Brauer–Wall group $BW_{\mathbb{C}}$ acting on a set of these two types is isomorphic to the cyclic group $\mathbb{Z}_2$. The cyclic structure of the group $BW_{\mathbb{C}} \simeq \mathbb{Z}_2$ may be represented on the following Trautman diagram (Fig. 3) by means of a transition $\mathbb{C}_n^+ \xrightarrow{h} \mathbb{C}_n$ (the round on the diagram is realized by an hour-hand). At this point, the type of algebra on the diagram is defined by an equality $n = h + 2r$, where $h \in \{0, 1\}$, $r \in \mathbb{Z}$.

It is obvious that a group structure over $\mathbb{C}_n$, defined by the group $BW_{\mathbb{C}} \simeq \mathbb{Z}_2$, immediately relates with a Modulo 2 periodicity of the complex Clifford algebras (Atiyah *et al.*, 1964; Karoubi, 1979): $\mathbb{C}_{n+2} \simeq \mathbb{C}_n \otimes \mathbb{C}_2$.

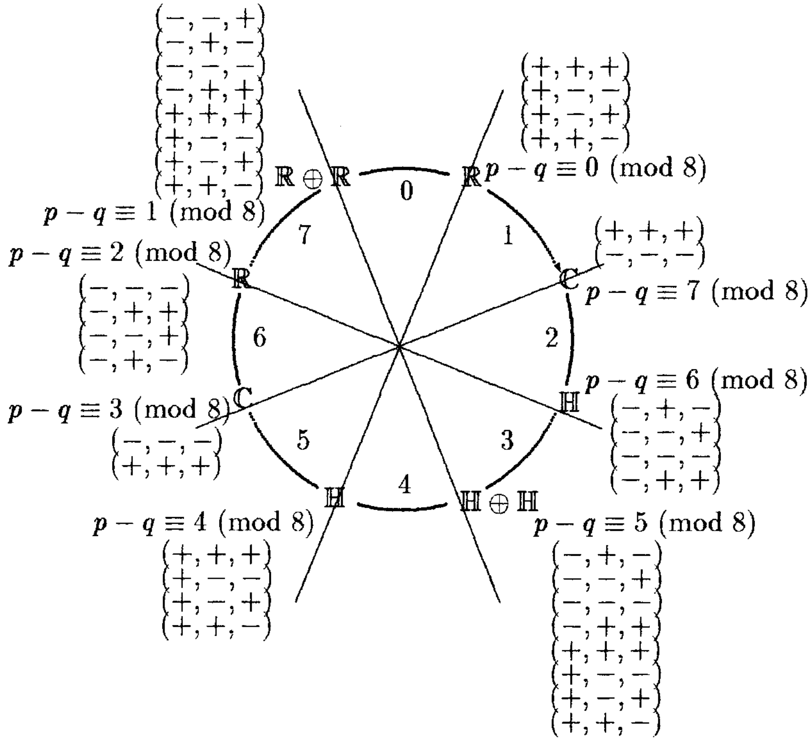

Fig. 2. The cyclic structure of the generalized group $BW_{\mathbb{R}}^{a,b,c}$.

From Theorem 2, it follows that the algebra $\mathbb{C}_{2m} \simeq \mathrm{M}_{2^m}(\mathbb{C})$ admits the automorphism group $\mathsf{Aut}_-(\mathbb{C}_{2m}) \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$ with the signature $(+, +, +)$ if m is even, and the group $\mathsf{Aut}_+(\mathbb{C}_{2m}) \simeq Q_4/\mathbb{Z}_2$ with the signature $(-, -, -)$ if m is odd. In connection with this, the second complex type $\mathbb{C}_{2m+1} \simeq \mathbb{C}_{2m} \oplus \mathbb{C}_{2m}$ also admits

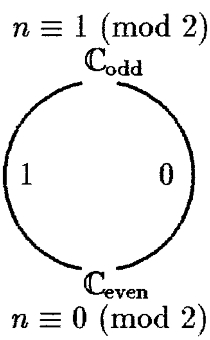

Fig. 3. The Trautman diagram for the Brauer–Wall group $BW_{\mathbb{C}} \simeq \mathbb{Z}_2$.

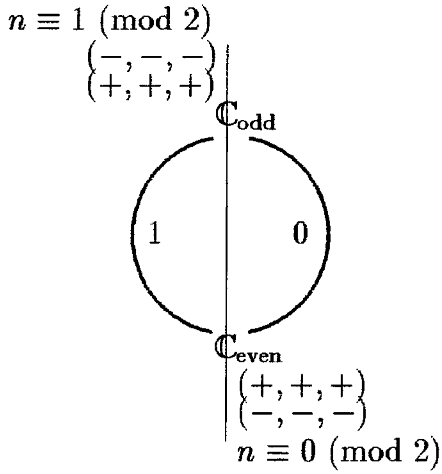

Fig. 4. The cyclic structure of the generalized group $BW_{\mathbb{C}}^{a,b,c}$.

both the previously mentioned automorphism groups. Therefore, if we take into account this relation, the cyclic structure of a generalized group $BW_{\mathbb{C}}^{a,b,c}$ would look as follows (Fig. 4). Both complex types $n \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$ form an axis of the second order; therefore, $BW_{\mathbb{C}}^{a,b,c} \simeq \mathbb{Z}_2$.

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REFERENCES

Abłamowicz, R. (1996). Clifford algebra computations with Maple. In *Proc. Clifford (Geometric) Algebras, Banff, Alberta Canada, 1995*. W. E. Baylis, ed., Birkhäuser, Boston, pp. 463–501.

Abłamowicz, R. (1998). Spinor representations of Clifford algebras: A symbolic approach, CPC thematic issue 'Computer Algebra in Physics Research.' *Physics Communications* **115**, 510–535.

Abłamowicz, R. (2000). CLIFFORD—Maple V package for Clifford algebra computations, Ver.4, `http://math.tntech.edu/rafal/cliff4/`.

Atiyah, M. F., Bott, R., and Shapiro, A. (1964). Clifford modules. *Topology* **3**(Suppl. 1), 3–38.

Berestetskii, V. B., Lifshitz, E. M., and Pitaevskii, L. P. (1982). *Quantum Electrodynamics Course of Theoretical Physics*, Vol. 4, 2nd edn., Pergamon Press, Oxford.

Blau, M. and Dąbrowski, L. (1989). Pin structures on manifolds quotiented by discrete groups. *Journal of Geometry and Physics* **6**, 143–157.

Budinich, P. and Trautman, A. (1987). An introduction to the spinorial chessboard. *Journal of Geometry and Physics* **4**, 363–390.

Budinich, P. and Trautman, A. (1988). *The Spinorial Chessboard*, Springer, Berlin.

Cahen, M., Gutt, S., and Trautman, A. (1995). Pin structures and the modified Dirac operator. *Journal of Geometry and Physics* **17**, 283–297.

Cahen M., Gutt, S., and Trautman, A. (1998). Pin structures and the Dirac operator on real projective spaces and quadrics. In *Clifford Algebras and Their Applications in Mathematical Physics*, K. Habetha, V. Dietrich, and G. Tank, eds., Kluwer Academic Publishers, 391–399.

Carlip, S. and De Witt–Morette, C. (1988). Where the sign of the metric makes a difference. *Physical Review Letters* **60**, 1599–1601.

Chamblin, A. (1994). On the obstructions to non-Cliffordian Pin structures. *Communications in Mathematical Physics* **164**, 67–87.

Chevalley, C. (1954). *The Algebraic Theory of Spinors*, Columbia University Press, New York.

Chevalley, C. (1955). The construction and study of certain important algebras. *Publications of Mathematical Society of Japan*, No. 1, Herald Printing, Tokyo.

Choquet-Bruhat, Y., De Witt-Morette, C., and Dillard-Bleick, M. (1982). *Analysis, Manifolds and Physics*, North-Holland, Amsterdam.

Dąbrowski, L. (1988). *Group Actions on Spinors*, Bibliopolis, Naples.

De Witt-Morette, C. and De Witt, B. (1990). The Pin groups in physics. *Physical Review D* **41**, 1901–1907.

De Witt-Morette, C. (1989). Topological obstructions to Pin structures. In *Geom. and Algebraic Aspects of Nonlinear Field Theory: Proc. Meet. Amalfi, May, 23–28 (1988)*, 113–118.

De Witt-Morette, C. and Gwo, Shang-Jr. (1990). *One Spin Group; Two Pin Groups*, Gauss Symposium, Gaussanium Inst.

De Witt-Morette, C., Gwo, Shang-Jr., and Kramer, E. (1997). *Spin or Pin?*, preprint; http://www.rel.ph.utexas.edu/Members/cdewitt/SpinOrPin1.ps.

Friedrich, Th. and Trautman, A. (1999). *Spin Spaces, Lipschitz Groups, and Spinor Bundles*, preprint SFB 288, N 362, TU-Berlin.

Hurwitz, A. (1923). Uber die Komposition der quadratischen Formen. *Math. Ann.* **88**, 1–25.

Karoubi, M. (1979). *K-Theory. An Introduction*, Springer-Verlag, Berlin.

Kirby, R. C. and Taylor, L. R. (1989). Pin structures on low-dimensional manifolds. In *London Math. Soc. Lecture Notes*, No. 151, Cambridge University Press, Cambridge.

Lipschitz, R. (1886). *Untersuchungen über die Summen von Quadraten*, Max Cohen und Sohn, Bonn.

Lounesto, P. (1981). Scalar products of spinors and an extension of Brauer–Wall groups. *Found. Phys.* **11**, 721–740.

Lounesto, P. (1993). Clifford algebras and Hestenes spinors. *Found. Phys.* **23**, 1203–1237.

Lounesto, P. (1997). *Clifford Algebras and Spinors*, Cambridge University Press, Cambridge.

Minkowski, H. (1909). Raum und Zeit, *Phys. Zs.* **10**, 104.

Miralles, D., Parra, J. M., and Vaz, Jr., J. (in press). Signature change and Clifford algebras, preprint math-ph/0003041; *Int. J. Theor. Phys.*

Porteous, I. R. (1969). *Topological Geometry*, van Nostrand, London.

Radon, J. (1922). Lineare Scharen orthogonaler Matrizen. *Abh. Math. Seminar Hamburg* **1**, 1–24.

Rashevskii, P. K. (1957). The theory of Spinors. (in Russian) *Uspekhi Mat. Nauk* **10**, 3–110 (1955); (in English) *Amer. Math. Soc. Transl. (Ser.2)* **6**, 1.

Salingaros, N. (1981). Realization, extension, and classification of certain physically important groups and algebras. *J. Math. Phys.* **22**, 226–232.

Salingaros, N. (1982). On the classification of Clifford algebras and their relation to spinors in n dimensions. *J. Math. Phys.* **23**(1).

Salingaros, N. (1984). The relationship between finite groups and Clifford algebras. *J. Math. Phys.* **25**, 738–742.

Varlamov, V. V. (1999). Fundamental automorphisms of Clifford algebras and an extension of Dąbrowski Pin groups. *Hadronic Journal* **22**, 497–533.

Wall, C. T. C. (1964). Graded Brauer groups. *J. Reine und Angew. Math.* **213**, 187–199.

Wigner, E. P. (1964). Unitary representations of the inhomogeneous Lorentz group including reflections. In *Group Theoretical Concepts and Methods in Elementary Particle Physics*, F. Gürsey, ed., Gordon and Breach, New York.